

LOGIC BLOG 2011

ABSTRACT. This year's logic blog has focussed on:

1. Demuth randomness
2. traceability
3. The connection of computable analysis and randomness
4. K -triviality in metric spaces.

CONTENTS

Part 1. Randomness, Kolmogorov complexity, and computability	2
1. Jan-May 2011: Demuth randomness, weak Demuth randomness, and the corresponding lowness notions	2
1.1. Basic definitions	2
1.2. Arithmetical Complexity	2
1.3. The diamond class of the non-weakly Demuth randoms	3
1.4. Lowness for weak Demuth randomness	4
1.5. BLR-traceability	4
1.6. Existence	5
1.7. BLR Traceability is Equivalent to Lowness for $\text{Demuth}_{\text{BLR}}$	6
2. The collection of weakly-Demuth-random reals is not Σ_3^0	7
Part 2. Traceability	8
3. Oct 2011: There are 2^{\aleph_0} -many $n^{1+\epsilon}$ -jump traceable reals	8
4. December 2011: A c.e. K -trivial which is not $o(\log x)$ jump traceable.	9
Part 3. Randomness and computable analysis	11
5. April 2011: Randomness and Differentiability	11
5.1. General thoughts	12
5.2. Denjoy alternative, even for partial functions	14
5.3. Denjoy randomness coincides with computable randomness	15
5.4. The Denjoy alternative for functions satisfying effectiveness notions weaker than computable	16
5.5. Sets of non-differentiability for single computable functions $f: [0, 1] \rightarrow \mathbb{R}$	22
5.6. Extending the results of [7] to higher dimensions.	23
5.7. The Lebesgue differentiation theorem	26
6. K -triviality and incompressibility in computable metric spaces	28
6.1. Background on computable metric spaces	28
6.2. K -triviality	28
6.3. K -trivial points in computable metric space	29
6.4. Existence of non-computable K -trivial points	30

6.5.	An equivalent local definition of K -triviality	30
6.6.	Local definition of K -triviality	31
6.7.	Incompressibility and randomness	35
Part 4.	Various	38
7.	Some new exercises on computability and randomness	38
	References	40

Part 1. Randomness, Kolmogorov complexity, and computability

1. JAN-MAY 2011: DEMUTH RANDOMNESS, WEAK DEMUTH RANDOMNESS, AND THE CORRESPONDING LOWNESS NOTIONS

This section contains news on Demuth randomness and weak Demuth randomness due to Nies and Kučera. They characterize arithmetical complexity and study the diamond class of no-weakly Demuth random.

The main result of this section is on lowness for (weak) Demuth randomness, due to Bienvenu, Downey, Greenberg, Nies, Turetsky. They characterize lowness for Demuth by a concept called BLR-traceability, which implies being in computably dominated \cap jump traceable (they can now show that it is a proper subclass). Paper submitted Feb. 2012.

1.1. Basic definitions. For a set $W \subseteq 2^{<\omega}$, we let

$$[W]^{\prec} = \{Z \in 2^{\mathbb{N}} : \exists n \ Z \upharpoonright_n \in W\},$$

the corresponding open class in Cantor space.

Definition 1.1. A Demuth test is a sequence of c.e. open sets $(S_m)_{m \in \mathbb{N}}$ such that $\forall m \ \lambda S_m \leq 2^{-m}$, and there is a function $f \leq_{\text{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^{\prec}$.

A set Z passes the test if $Z \notin S_m$ for almost every m . We say that Z is Demuth random if Z passes each Demuth test.

If we apply the usual passing condition for tests, we obtain the following notion.

Definition 1.2. We say that Z is weakly Demuth random if for each Demuth test $(S_m)_{m \in \mathbb{N}}$ there is an m such that $Z \notin S_m$.

Remark 1.3. Recall that $f \leq_{\text{wtt}} \emptyset'$ if and only if f is ω -c.e., namely, $f(x) = \lim_t g(x, t)$ for some computable function g such that the number of changes $g(x, t) \neq g(x, t-1)$ is computably bounded in x . Hence the intuition is that we can change the m -component S_m a computably bounded number of times. We will denote by $S_m[t]$ the version of the component S_m that we have at stage t . Thus $S_m[t] = [W_{g(m,t)}]^{\prec}$ where g is understood to be a computable approximation of f as above.

1.2. Arithmetical Complexity. Kučera and Nies worked in Prague in May. They proved the following.

Proposition 1.4. (i) Weak Demuth randomness is Π_3^0 .
(ii) Demuth randomness is not Π_3^0 .

In [15] they had shown that both notions are Π_4^0 .

Proof. (i) Let $(\Gamma_e, h_e)_{e \in \mathbb{N}}$ be an effective listing of pairs of Turing functionals and use bounds (pc functions), such that if $p = \Gamma_e^X(m) \downarrow$ then $\lambda[W_p]^\prec \leq 2^{-m}$. If $\Gamma_e^{\emptyset'}$ is total with use bound h_e (also total) then this determines a Demuth test according to Definition 1.1. Conversely, we obtain all Demuth tests that way. Then Z is not weakly Demuth random iff

$$\exists e \forall m \forall s \exists t > s [p = \Gamma_e^{\emptyset'}(m)[t] \downarrow \text{ with use bound } h_{e,t}(m) \wedge Z \in [W_{p,t}]^\prec].$$

For, if the pair $(\Gamma_e^{\emptyset'}, h_e)$ does not determine a Demuth test then the condition for e fails. This shows (i).

(ii) Otherwise, the class of non-Demuth random ML-random sets is a Σ_3^0 null class, so there must be a c.e. noncomputable set A below all of them. This contradicts the fact that there is a high ML-random Δ_2^0 set $Z \not\leq_T A$; this set is not GL_1 and hence not Demuth random. (See [15, Thm 3.2] for a somewhat stronger result that even implies the existence of a weakly Demuth random set Z .)

□

1.3. The diamond class of the non-weakly Demuth randoms.

Theorem 1.5 (Kučera and Nies). *Let A be a c.e. set. Then*

*A is Turing below all ML-random but non-weak Demuth random sets \Leftrightarrow
 A is strongly jump traceable.*

Proof. (\Rightarrow) An ω -c.e. set is not weakly Demuth random. If A is below all ω -c.e. ML-randoms then A is already s.j.t. by a result of [13].

(\Leftarrow) Suppose $(S_m)_{m \in \mathbb{N}}$ is a Demuth test with $S_m \supseteq S_{m+1}$, . We need to show that $A \leq_T Y$ for any ML-random set Y such that $Y \in \bigcap_m S_m$.

As in Remark 1.3 we have $S_m[t] = [W_{g(m,t)}]^\prec$ where g is a computable binary function with a computably bounded number of changes $g(m, t) \neq g(m, t-1)$. We may assume that $S_m[t] \supseteq S_{m+1}[t]$ for each m, t .

For $m < s$ let $r(m, s)$ be the maximum e such that $g(i, t)$ is stable for stages t , $m < t \leq s$, for all $i \leq e$. Let

$$G_m = \bigcup_{s > m} [W_{g(r(m,s), s)}]^\prec$$

Informally, at stage s , the Σ_1^0 set G_m copies the version S_e for e largest that hasn't changed since m . If it does change G_m takes the (larger) version S_{e-1} etc. till this stops.

Clearly $\bigcap_e S_e$ is contained in the Π_2^0 null class $\bigcap_m G_m$. Now consider the usual Hirschfeldt-Miller cost function

$$c(m, s) = \lambda G_{m,s}.$$

Recall that if a set A obeys c then A is below each ML-random set $Y \in \bigcap_m G_m$ by [22, Thm. 5.3.x].

This cost function c is benign. For let $c(x, s) > 2^{-e}$. Then $r(x, s) < e$, so $g(i, t)$ has changed for some $i \leq e$ between stages x and s . So benignity follows from the hypothesis that (S_e) is a Demuth test.

Now, by [14], the c.e. strongly jump traceable set A obeys c . □

Let \mathcal{C} be the class of non-weakly Demuth random sets. Then the theorem says that \mathcal{C}^\diamond coincides with the strongly jump traceable c.e. sets. We also considered the question whether there is a single weakly Demuth random above all c.e. strongly jump traceables. If not, then this would mean that the non-weakly Demuth random sets form the largest class \mathcal{C} with $\mathcal{C}^\diamond =$ the c.e. strongly jump traceables, where for a class \mathcal{C} only the intersection with MLR counts.

1.4. Lowness for weak Demuth randomness. For details on the following summary, see our forthcoming paper [3]. Z is weakly Demuth *by* an oracle set A if it passes every weak A -Demuth test where the number of version changes is computably bounded.

Proposition 1.6. *The following are equivalent for a set A :*

- (i) *Every weak Demuth random is weak Demuth random by A .*
- (ii) *A is K -trivial.*

Proof. The implication (i) \Rightarrow (ii) follows from a result of Bienvenu and Miller [4].

Other direction by Claim 5.7 of Nies' paper "c.e. sets below random sets" [23]. \square

Theorem 1.7. *The only sets that are low for weak Demuth randomness are the computable sets.*

Since each K -trivial is Δ_2^0 , by Proposition 1.6 it suffices to prove the following.

Lemma 1.8. *Let A be a set computing a function f which is not dominated by any computable function. Then there is a weakly Demuth random Z which is not weakly Demuth random relative to A .*

1.5. BLR-traceability.

Definition 1.9. *A BLR trace is a sequence $(T_n)_{n \in \mathbb{N}}$ such that $T_n = W_{r(n)}$ where r is an ω -c.e. function. Let h be an order function. We say h is a bound for (T_n) if $|T_n| \leq h(n)$ for each n .*

Definition 1.10. *We say that A is BLR traceable if there is an order function h such that each ω -c.e. by A function f has a BLR trace with bound h .*

As usual, the choice of h doesn't matter. Also, if A is computably dominated then we can take equivalently $T_n = D_{p(n)}$ for some ω -c.e. function p .

Fact 1.11. *Jump traceable & superlow is equivalent to BLR traceable with bound 1.*

Fact 1.12. *Jump traceable & c.e. implies BLR traceable with bound 1.*

These are both by Cole/Simpson, and because having a BLR trace with bound 1 means being ω -c.e.

Proposition 1.13. *BLR traceable with constant bound > 1 does not imply ω -c.e.*

Fact 1.14. *BLR traceable implies jump traceable.*

This is so because the function $f(x) = J^A(x)$ if defined, 0 otherwise is ω -c.e. by A .

Theorem 1.15. *There is an ω -c.e. set which is jump traceable but not BLR traceable.*

Theorem 1.16. *There is jump traceable and computably dominated set that is not BLR traceable.*

Proofs are in the paper.

1.6. Existence.

Theorem 1.17. *There is a special Π_1^0 class of BLR traceable sets.*

Proof. Let $\{(\Gamma_e, g_e)\}_{e \in \omega}$ be an enumeration of (partial) ω -c.e. by oracle functions. So the following hold:

- g_e is a partial computable function which converges on an initial segment of ω ;
- Γ_e^X is total for every oracle X ;
- $|\{t \mid \Gamma_e^X(n, t) \neq \Gamma_e^X(n, t+1)\}| < g_e(n)$ for every n such that $g_e(n) \downarrow$ and every oracle X .

We let $f_e^X(n) = \lim_t \Gamma_e^X(n, t)$.

We build a Π_1^0 -class \mathcal{P} and computable BLR traces $\{(T_n^e)_{n \in \omega}\}_{e \in \omega}$ with bound 2^n . For all i , we have the requirement:

R_i : ϕ_i is not a computable description of a real in \mathcal{P} .

For every pair (e, n) with $e < n$, we have the requirement:

$Q_{e,n}$: If $g_e(n) \downarrow$, $f_e^X(n) \in T_n^e$ for all reals $X \in \mathcal{P}$.

Clearly these requirements will suffice to prove the result.

Each strategy will receive from the previous strategy some finite collection of strings $\{\alpha_j\}$ all of the same length, and it will create some finite extensions $\{\beta_k\}$ (all of the same length) such that for every j there is at least one k such that $\alpha_j \subseteq \beta_k$, and all reals in $\bigcup_k [\beta_k]$ meet the strategy's requirement.

R_i -strategies will initially define exactly two β_k for every α_j , but may later remove one.

$Q_{e,n}$ -strategies will define exactly one β_k for every α_j . They may need to redefine β_k some finite number of times, but each new definition will be an extension of the previous.

At the end of every stage s , we let $\{\hat{\beta}_k\}$ be the outputs of the last strategy to act at stage s , and define the tree P_s to be all strings comparable with one of the $\hat{\beta}_k$. \mathcal{P} will be $\bigcap_s [P_s]$.

Description of R_i -strategy:

This is the standard non-computability requirement on a tree:

- (1) Let $\{\alpha_j\}_{j < m}$ be the output of the previous strategy. For every $j < m$, let $\beta_{j,0} = \alpha_j 0$ and $\beta_{j,1} = \alpha_j 1$.
- (2) Wait for $\phi_i(|\alpha_0|)$ to converge; while waiting, let the outputs be $\{\beta_{j,0}, \beta_{j,1}\}_{j < m}$.

- (3) When $\phi_i(|\alpha_0|)$ converges. . .
- . . . if $\phi_i(|\alpha_0|) = 0$, let the outputs be $\{\beta_{j,1}\}_{j < m}$.
 - . . . if $\phi_i(|\alpha_0|) \neq 0$, let the outputs be $\{\beta_{j,0}\}_{j < m}$.

Description of $Q_{e,n}$ -strategy:

Until $g_e(n)$ converges, this strategy takes no action. We ignore for the moment the computable bound on the number of times the index of T_e^n changes.

Let $\{\alpha_j\}_{j < m}$ be the output of the previous strategy. We will keep several values to assist the strategy: ℓ_s will be the number of times the output has been redefined by stage s ; $c_s(j)$ will be the current guess for $f_e(n)$ on an extension of α_j . We initially have $\ell_s = 0$ and $c_s(j) = -1$ for all j . Unless otherwise defined, $\ell_{s+1} = \ell_s$ and $c_{s+1}(j) = c_s(j)$.

For every j , let $\beta_j^0 = \alpha_j$. We initially let the outputs be $\{\beta_j^0\}_{j \in \omega}$ and define $T_n^e = \emptyset$. We run the following strategy, where s is the current stage:

- (1) Wait for a string $\gamma \in P_s$ with γ extending one of the $\beta_j^{\ell_s}$ and $\Gamma_e^\gamma(n, s) \neq c_s(j)$.
- (2) When such a string is found for $\beta_j^{\ell_s}$:
 - (a) Define $\beta_j^{\ell_s+1} = \gamma$ and $c_{s+1}(j) = \Gamma_e^\gamma(n, s)$.
 - (b) For every $k < m$ with $k \neq j$, choose $\beta_k^{\ell_s+1}$ extending $\beta_k^{\ell_s}$ of the same length as $\beta_j^{\ell_s+1}$.
 - (c) Redefine $T_n^e = \{c_{s+1}(k) \mid k < m\}$.
 - (d) Define $\ell_{s+1} = \ell_s + 1$.
- (3) Return to Step 1.

Full construction:

We make the assumption that for every s , there is precisely one pair (e, n) with $e < n$ and $g_{e,s+1}(n) \downarrow$ but $g_{e,s}(n) \uparrow$. We give the $Q_{e,n}$ -strategies priority based on the order in which the $g_e(n)$ converge: if $g_e(n)$ converges before $g_{e'}(n')$, then the $Q_{e,n}$ -strategy has stronger priority than the $Q_{e',n'}$ -strategy. If $g_e(n)$ never converges, then $Q_{e,n}$ never has a priority, but this is fine because it never acts.

We prioritize the R_i -strategies based on the priorities of the $Q_{e,n}$ -strategies: R_i has weaker priority than $R_{i'}$ for any $i' < i$, and also weaker priority than any $Q_{e,n}$ -strategy with $n \leq i$. It is given the strongest priority consistent with these restrictions.

Since we only consider $n > e \geq 0$, the R_0 -strategy will always have strongest priority. It receives $\alpha_0 = (i)_{i \in \mathbb{N}}$ as the “output of the previous strategy”.

At stage s , let (e, n) be the pair such that $g_e(n)$ has newly converged. We initialise R_n and all strategies which had weaker priority than R_n . The priorities of the various R_i are then redetermined. We then let all Q -strategies with priorities and all R_i -strategies with $i < s$ act, in order of priority.

Whenever a strategy redefines its output, all weaker priority strategies are initialised.

□

1.7. BLR Traceability is Equivalent to Lowness for Demuth_{BLR}.

We say that a set Z is Demuth random *by* an oracle set A (Demuth_{BLR}

A for short) if it passes every A -Demuth test where the number of version changes is computably bounded. A is low for $\text{Demuth}_{\text{BLR}}$ if every Demuth random is already $\text{Demuth}_{\text{BLR}}$ A .

Theorem 1.18. *The following are equivalent for a set A .*

- (i) A is BLR traceable.
- (ii) A is low for $\text{Demuth}_{\text{BLR}}$.

Corollary 1.19. *There exists a perfect class of sets which are all low for Demuth randomness.*

Proof. First observe that any computably dominated set which is low for $\text{Demuth}_{\text{BLR}}$ is already low for Demuth randomness. Now by a result of Martin/Miller (1968), the Π_1^0 -class obtained in Theorem 1.17 contains a perfect subclass of sets which are computably dominated. By the previous theorem, this class is as required. \square

2. THE COLLECTION OF WEAKLY-DEMUTH-RANDOM REALS IS NOT Σ_3^0

By Yu Liang.

This is a result corresponding to Proposition 1.4.

Lemma 2.1. *Given any recursive tree $T \subseteq 2^{<\omega}$ so that $[T]$ only contains Martin-Löf random reals, there is a weakly-Demuth-test $\{W_{f(i)}\}_{i \in \omega}$ so that for any $\sigma \in 2^{<\omega}$, if $[\sigma] \cap [T] \neq \emptyset$, then $[\sigma] \cap [T] \cap (\bigcap_i W_{f(i)}) \neq \emptyset$.*

Proof. The proof is quite similar to the one by Yu for weak-2-randomness.

Given a recursive tree $T \subseteq 2^{<\omega}$ so that $[T]$ only contains Martin-Löf random reals, we try to build a weakly-Demuth-test $\{W_{f(i)}\}_{i \in \omega}$ so that $W_{f(i)}$ densely meets $[T]$ for every i . Suppose $\mu(T) > 2^{-r}$ for some number r .

To build $W_{f(i)}$, for each σ , we try to search some $\tau \succ \sigma$ with $|\tau| = |\sigma| + \lceil \sigma \rceil + i + 1$ so that $[\tau] \cap [T] \neq \emptyset$ where $\lceil \sigma \rceil$ is the Gödel number of σ . We put such τ 's into $W_{f(i,0)}$. The searching way is “from left to right”. In the most lucky case, we don't make mistake, then $W_{f(i,0)}$ densely meets $[T]$ and $\mu(W_{f(i,0)}) \leq 2^{-i-1}$. But of course we may make mistakes. Note that “making mistakes” is a Σ_1^0 -sentence. Once we find the measure of mistakes greater than 2^{-i-1} , we change $W_{f(i,0)}$ to be $W_{f(i,1)}$. By some tricks, we can ensure not to make the same mistakes. This means the times of “making big mistakes” is no more than 2^{i+1} times.

So $f(i)$ change at most 2^i times and $\mu(W_{f(i)}) \leq 2^{-i}$. \square

The left part is the exactly same as weak-2-randomness case.

Let $\mathbb{P} = (\mathbf{T}, \leq)$ be partial ordering so that every $T \in \mathbf{T}$ is a recursive tree and only contains Martin-Löf random reals. $T_1 \leq T_2$ if and only if $T_1 \subseteq T_2$. Clearly every \mathbb{P} -generic real is weakly-2-random and so weakly-Demuth-random.

Lemma 2.2. *Given any Π_2^0 set G only containing weakly-Demuth-random reals, the set $\mathcal{D}_G = \{P \in \mathbf{T} \mid P \cap G = \emptyset\}$ is dense.*

Proof. Immediately from Lemma 2.1. \square

So for any Σ_3^0 set H only containing weakly-Demuth-random reals, by Lemma 2.2, any sufficiently \mathbf{P} -generic real doesn't belong to H .

I don't know whether the collection of Demuth-random reals is Σ_3^0

Part 2. Traceability

3. OCT 2011: THERE ARE 2^{\aleph_0} -MANY $n^{1+\epsilon}$ -JUMP TRACEABLE REALS

By Yu Liang. This is joint work with Denis Hirschfeldt.

Definition 3.1. Let h be an order function. We say that a real x is h -jump traceable if for each Turing functional Φ , there is an h -bounded c.e. trace $(U_n)_{n \in \mathbb{N}}$ such that Φ^x total implies $\Phi^x(n) \in U_n$ for a.e. n .

Theorem 3.2. Let ϵ be an arbitrary small positive rational. Then there are 2^{\aleph_0} -many $n^{1+\epsilon}$ -jump traceable reals.

We construct a perfect tree T so that there is a uniformly r.e. sequence $\{U_e\}_{e \in \omega}$ such that

- For every e , $|U_e| \leq e^{1+\epsilon}$;
- for any $x \in [T]$ and any e , if Φ_e^x is total, then $\Phi_e^x(n) \in U_n$ for any $n \geq e$.

First we show that for a single index e , there is a uniformly r.e. sequence $\{U_e\}_{e \in \omega}$ such that

- For every e , $|U_e| \leq e^{1+\epsilon}$;
- for any $x \in [T]$, if Φ_e^x is total, then $\Phi_e^x(n) \in U_n$ for any n .

Let $\delta(n)$ be a computable increasing function so that

$$(\delta(n))^{1+\epsilon} > 2^{n+1} \sum_{i \leq n} \delta(i).$$

We build an embedding function $T : 2^{<\omega} \rightarrow 2^\omega$ by approximation.

At stage 0, T_0 is identity.

At stage $s+1$, if there is some $m \in [\delta(n), \delta(n+1))$ and some finite string σ , where we may assume that $|\sigma| > n$, so that $\Phi_e^{T_s(\sigma)}(m) \downarrow$ at stage s . Let $T_{s+1}[\sigma \upharpoonright n]$ be $T_s[\sigma]$ and move other values corresponded to remain T_{s+1} to be an embedding function. In other words, we kill all the branching nodes up to $T_s(\sigma)$ extending $T_s(\sigma \upharpoonright n)$ to narrowing the possible values of Φ_e . Put $\Phi_e^{T_s(\sigma)}(m)$ into U_m .

The intuition behind the construction is to make all the values between $[\delta(n), \delta(n+1))$ be "on some single node".

By the finite injury, $\lim_{s \rightarrow \infty} T_s$ exists.

Note that for each $m \in [\delta(n), \delta(n+1))$, we put at most $\delta(n) \cdot 2^{n+1}$, which is not greater than $m^{1+\epsilon}$, many values into U_m .

For the general case, there is no essential difference. Just let δ be even faster so that we may narrow the possible branches.

Conjecture 3.3. There are only countably many id-jump traceable reals.

4. DECEMBER 2011: A c.e. K -TRIVIAL WHICH IS NOT $o(\log x)$ JUMP TRACEABLE.

By Turetsky. Paper accepted in Information Processing Letters.

Lemma 4.1. *If h is an order with $\sum_{x=0}^{\infty} 2^{-d \cdot h(x)} = \infty$ for all $d > 0$, then for any $c > 0$, there is an $n > 0$ with $h(2^{cn}) \leq n$.*

Proof. Fix a c , and suppose $h(2^{cn}) > n$ for all n . Then $h(x) > \frac{1}{c} \log x$ for all x , and thus $\sum_{x=0}^{\infty} 2^{-2c \cdot h(x)} < \infty$, contrary to hypothesis. \square

Theorem 4.2. *There is a c.e. K -trivial set A which is not jump traceable at any order h with $\sum_{x=0}^{\infty} 2^{-d \cdot h(x)} = \infty$ for all $d > 0$. In particular, it is not jump traceable at any order $h \in o(\log x)$.*

Proof. We make A K -trivial in a standard fashion: fix W a c.e. operator such that $\lambda([W^X]) \leq 1/2$ and $[W^X]$ contains all X -randoms for all oracles X (e.g., $[W^X]$ is an element of the universal oracle ML-test); we shall construct a c.e. set V with $[W^A] \subseteq [V]$ and $\lambda([V]) < 1$. Our construction of V is the obvious one: whenever we see a string $\sigma \in W^A[s]$, we enumerate σ into V . We will ensure that $\lambda([V]) < 1$ by appropriate restraint on A .

Let h^e be an enumeration of all (partial) orders, and let $(T_k^e(x)_i)_{i \in \mathbb{N}} \in \omega$ be an enumeration of all h^e bounded c.e. traces. We construct a partial A -computable function f^A , and (assuming h^e satisfies the hypothesis of the theorem) make A not jump traceable at order h^e by meeting the following requirements:

$$P_k^e: (\exists x)[f^A(x) \downarrow \ \& \ f^A(x) \notin T_k^e(x)].$$

We partition the domain of f^A into infinitely many sets B^e , and work to meet requirements for h^e on B^e . However, our choice of pairing function matters: each B^i must be an arithmetic progression. So we let $B^e = \{x \cdot 2^{e+1} + 2^e \mid x \in \omega\}$.

The basic P_k^e -strategy is straightforward. Choose an $x \in B^e$, wait until $h^e(x) \downarrow$, and then define $f^A(x)$ to a large value with large use. Wait until $f^A(x) \in T_k^e(x)$. Change A below the use and redefine $f^A(x)$ to a large value. Eventually we win, since $T_k^e(x)$ has bounded size.

The complication comes in the interaction between positive requirements and ensuring that $\lambda([V]) < 1$ — redefining $f^A(x)$ may cause measure to leave $[W^A]$. As long as the total measure which leaves $[W^A]$ over the course of the construction is strictly less than $1/2$, we are fine. P_k^e must be careful to only contribute a small fraction of that.

Suppose that for some constant c , P_k^e has claimed for its future use $2^{c \cdot n}$ many elements of B^e on which h^e takes value less than n , for some n . P_k^e will follow the basic strategy on the first of these elements, but will discard this and choose a new one if the cost of clearing the computation rises too high. What is “too high”? This will depend on how much progress P_k^e has made on the current witness. Initially (if P_k^e has not yet cleared any computations on this box), P_k^e will discard the box if the measure lost by clearing the computation rises beyond $2^{-c \cdot n}$. Each time P_k^e completes a loop of the basic strategy (that is, each time it clears a computation and the current n -box is promoted), its threshold is multiplied by 2^{c-1} . So if the box has been promoted m times, P_k^e will discard it if the measure lost by clearing the

computation rises beyond $2^{-c \cdot n + (c-1) \cdot m}$. When it discards a box and begins working with the next one, P_k^e 's threshold returns to $2^{-c \cdot n}$.

Now, let us analyze this strategy in the absence of action for other positive requirements. Assume that $c > 1$. Because uses are always chosen large, the measure which restrains a given n -box of P_k^e is always disjoint from the measure which restrains a different n -box. So if x_1, \dots, x_l are the n -boxes P_k^e is restrained on, and box x_i is restrained after m_i promotions, we know

$$\sum_{i=1}^l 2^{-c \cdot n + (c-1) \cdot m_i} \leq 1/2.$$

Since $m_i \geq 0$, we know $l \leq 2^{c \cdot n}$. Actually, since the measure which restrains x_i must be strictly more than the threshold, we know that $l < 2^{c \cdot n}$.

Now, all the promotions of x_i caused the discarding of at most

$$\sum_{j=0}^{m_i-1} 2^{-c \cdot n + (c-1) \cdot j} \leq 2^{-c \cdot n + (c-1) \cdot m_i - c + 2}$$

measure. So the total measure discarded by all the x_i is at most

$$\sum_{i=1}^l 2^{-c \cdot n + (c-1) \cdot m_i - c + 2} = 2^{-c+2} \sum_{i=1}^l 2^{-c \cdot n + (c-1) \cdot m_i} \leq 2^{-c+1}.$$

Since $l < 2^{c \cdot n}$, we know there is some x_{l+1} which is never restrained, and thus causes the P_k^e strategy to succeed. Now, let us analyze the total measure discarded by the promotions of x_{l+1} . It is at most

$$\sum_{j=0}^n 2^{-c \cdot n + (c-1) \cdot j} \leq 2^{-c \cdot n + (c-1) \cdot n + 1} = 2^{-n+1}.$$

So the total discarded measure caused by P_k^e is at most $2^{-n+1} + 2^{-c+1}$.

We now integrate all the positive strategies via finite injury. When each P_k^e strategy is initialised, it chooses a large integer r . P_k^e then searches for $2^{c \cdot n}$ many unclaimed elements of B^e with $h^e \leq n$ on these elements and $2^{-n+1} + 2^{-c+1} < 2^{-(r+3)}$. By the previous lemma, if h^e satisfies the hypothesis of the theorem, since B^e is an arithmetic progression, P_k^e will eventually find such elements. It then begins running the above strategy. Whenever a higher priority strategy enumerates an element into A , P_k^e is initialised, choosing a new r and searching for new elements to work on. By the usual finite injury argument, every strategy eventually succeeds. Further, the lost measure is at most $\sum_{r=0}^{\infty} 2^{-(r+3)} = 1/4$. So $\lambda([V]) \leq 3/4$. \square

Part 3. Randomness and computable analysis

5. APRIL 2011: RANDOMNESS AND DIFFERENTIABILITY

by Nies (mainly), Bienvenu, Hoelzl, Turetsky and others.

Brattka, Miller and Nies have submitted their paper entitled Randomness and Differentiability [7]. The main thesis of the paper is that algorithmic randomness of a real is equivalent to differentiability of effective functions at the real. It goes back to earlier work of Demuth, for instance [8], and Pathak [24].

For most major algorithmic randomness notions, one can now provide a class of effective functions on the unit interval so that

(*) a real $z \in [0, 1]$ satisfies the randomness notion \iff
each function in the class is differentiable at z .

The matching between algorithmic randomness notions and classes of effective functions is summarized in Figure 1.

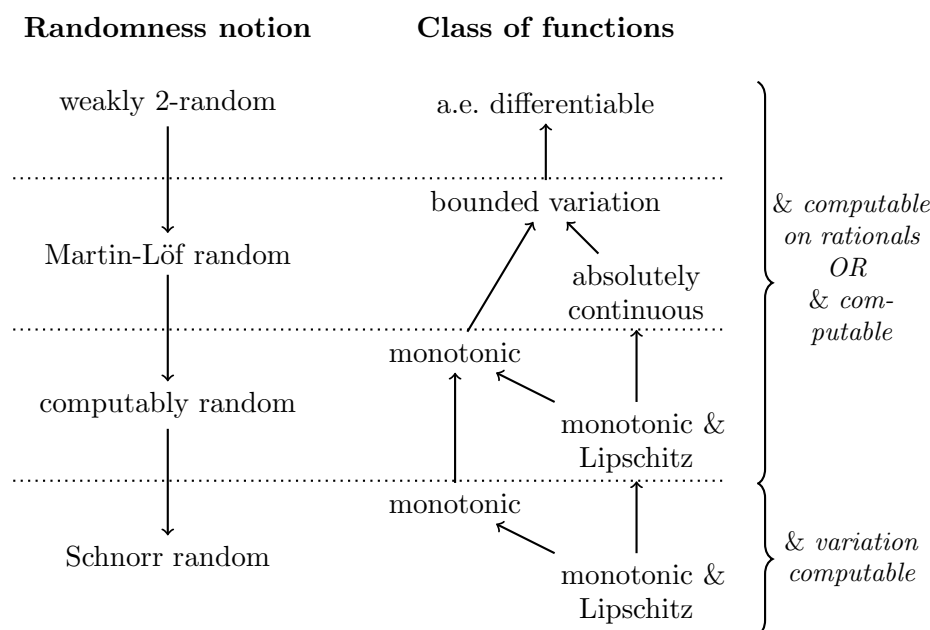


FIGURE 1. Randomness notions matched with classes of effective functions defined on $[0, 1]$ so that (*) holds

Groups currently working in this area include

- Freer, Kjos-Hanssen, Nies: computable Lipschitz function. In prep.
- J. Rute (Carnegie Mellon, student of J. Avigad): higher dimensions, measures. Characterized Schnorr randomness. In preparation.
- Pathak, Rojas and Simpson: Characterized Schnorr randomness. Submitted Nov 2011.
- Kenshi Myabe (Kyoto U): a view via function tests (integral tests). Characterized Schnorr randomness, Kurtz “randomness”. Submitted Nov 2011.

5.0.1. *Notation.* For a function f , the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a, b) = \frac{f(a) - f(b)}{a - b}$$

Recall that if z is in the domain of f then

$$\begin{aligned}\overline{D}f(z) &= \limsup_{h \rightarrow 0} S_f(z, z + h) \\ \underline{D}f(z) &= \liminf_{h \rightarrow 0} S_f(z, z + h)\end{aligned}$$

$f'(z)$ exists just if those are equal and finite.

5.1. General thoughts. Why do algorithmic randomness notions for reals match so well with analytic notions for functions in one variable? The functions have to be computable (or even computable in the variation norm), but then, most functions defined on the unit interval that arise in analysis, such as e^x or $\ln \sqrt{x+1}$, are computable. (If the derivative is computable the function is even variation computable.)

Analytic notions on functions have been studied systematically by Lebesgue [16] and even earlier. Algorithmic randomness notions have been studied for 45 years, starting with [17] (see [20] for the Russian side of the story).

5.1.1. *How about matching the remaining randomness notions?*

Question 5.1. *Can one match the notions of partial computable randomness and permutation randomness (see [22, Ch. 7]) with a class of real functions in the sense of (*)?*

First of all we would like a proof that these notions are actually about real numbers (not bit sequences). Just like for computable randomness [7, Section 4] one would need to show that they are independent of the choice of a base (which is 2 in the definition).

Question 5.2. *How about matching 2-randomness? Demuth randomness?*

Demuth's paper [9] went a long way towards answering the second part of the question: at a Demuth random real z , the Denjoy (French, 1907) alternative (DA) holds for each Markov computable (=constructive) function f . See Subsection 5.2 for definitions, and Subsection 5.4 for a result stronger than Demuth's.

5.1.2. *Are there function notions from analysis that can't be matched?*

This remains open. The analytic notions in Figure 1 seem to be the major ones.

5.1.3. *The case of ML-randomness.* Recall that a function $f: [0, 1] \rightarrow \mathbb{R}$ is of *bounded variation* if

$$\infty > \sup \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|,$$

where the sup is taken over all partitions $t_1 < t_2 < \dots < t_n$ in $[0, 1]$.

The characterization of ML-randomness is via differentiability of computable functions of bounded variation. A direct proof is probably in Demuth [8]. In [7] we get the harder direction \Rightarrow in (*) above out of the case for computable randomness via the low basis theorem.

A direct proof for \Rightarrow is not available currently. Here is a simple proof of a weaker statement:

Fact 5.3. *Let z be ML-random. If f is a computable function of bounded variation, then $\overline{D}f(z) < \infty$.*

Proof. We may assume that the variation $\text{Var}(f) \leq 1$. A dyadic n -interval inside $[0, 1]$ has the form $(i2^{-n}, k2^{-n})$ where $i < k$ are naturals. Let G_r be the union of all intervals (x, y) such that $S_f(x, y) > 2^r$ and that are dyadic n intervals for some n . We claim that $\lambda G_r \leq 2^{-r}$. For let $G_{r,n}$ be the union of dyadic n -intervals in G_r . Consider the partition consisting of the endpoints of the maximal dyadic n -intervals contained in $G_{r,n}$. Since $V(f) \leq 1$ we can conclude that $\lambda G_{r,n} \leq 2^{-r}$. Since $G_r = \bigcup_n G_{r,n}$ this shows $\lambda G_r \leq 2^{-r}$.

Now clearly (G_r) is uniformly c.e., hence a ML-test. If $\overline{D}f(z) = \infty$ then (because f is continuous) z fails this test. \square

5.1.4. *A simpler proof of a special case of [7, Theorem 4.1(iii) \rightarrow (ii)].* For a nondecreasing function $f: [0, 1] \rightarrow \mathbb{R}$ and a real r let M_r^f be the (dyadic) martingale associated with the slope S_f evaluated at intervals of the form $[r + i2^{-n}, r + (i+1)2^{-n}]$. Solecki, combining ideas from his paper [19] with the middle thirds lemma [7, 2.5], has shown the following (personal communication):

Theorem 5.4. *Suppose f is a nondecreasing function which is not differentiable at $z \in [0, 1]$. Suppose that $\underline{D}f(z) = 0$, or $\overline{D}f(z)/\underline{D}f(z) > 72$. Then one of the martingales $M^f, M_{1/3}^f$ does not converge on z .*

Proof. For an interval L with endpoints a, b we write $S(L)$ for $S_f(a, b)$. Note that $S(I \cup J) \leq \max\{S(I), S(J)\}$ for non-disjoint intervals I, J .

For $m \in \mathbb{N}$ let \mathcal{D}_m be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where $k \in \mathbb{Z}$. Let \mathcal{D}'_m be the set of intervals $(1/3) + I$ where $I \in \mathcal{D}_m$.

The underlying geometric fact is simple:

Fact 5.5. *Let $m \geq 1$. If $I \in \mathcal{D}_m$ and $J \in \mathcal{D}'_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.*

To see this: assume that $(k2^{-m} - (p2^{-m} + 1/3) < 1/(3 \cdot 2^m)$. This yields $3k - 3p - 2^m)/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$, and hence $3|2^m$, a contradiction.

Claim 5.6. *Let $z \in I \cap J$ for intervals $I \in \mathcal{D}_m, J \in \mathcal{D}'_m$. Suppose $z \in L$ for some interval L of length d , where $2^{-m}/3 \geq d \geq 2^{-m-1}/3$. Then*

$$(1) \quad S(L)/12 \leq \max\{S(I), S(J)\}.$$

Clearly by Fact 5.5 we have $L \subseteq I \cup J$. Let $I \cup J = [a, b]$. Since f is nondecreasing and $b - a \leq 12|L|$, we have

$S(L) \leq (f(b) - f(a))/|L| \leq 12S(I \cup J) \leq 12 \max\{S(I), S(J)\}$. This shows the claim.

We now give a dual claim where we need the middle thirds.

Claim 5.7. *Let $z \in I \cap J$ for intervals $I \in \mathcal{D}_m, J \in \mathcal{D}'_m$. Suppose z is in the middle third of some interval L of length d , where $2^{-m+1} \geq d/3 \geq 2^{-m}$. Then*

$$(2) \quad 6S(L) \geq \max\{S(I), S(J)\}.$$

We only prove it for I . Note that $I \subseteq L$ because z is in the middle third of L . Since $d \leq 6|I|$, and f is nondecreasing, we obtain $6S(L) \geq S(I)$ as required. This shows the claim.

Now we argue similar to [7]. By the hypothesis that $f'(z)$ does not exist and the middle thirds lemma [7, 2.5], we can choose rationals $\tilde{\beta} < \tilde{\gamma}$ such that $\tilde{\gamma}/\tilde{\beta} > 72$ and

$$\begin{aligned} \tilde{\gamma} &< \limsup_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \wedge z \in (x, y)\}, \\ \tilde{\beta} &> \liminf_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \wedge z \in \text{middle third of } (x, y)\} \end{aligned}$$

By definition we can choose arbitrarily short intervals L containing z such that $S(L) \geq \tilde{\gamma}$, and arbitrarily short intervals L containing z in their middle third such that $S(L) \leq \tilde{\beta}$. Then, by Claim 5.6, for one of the \mathcal{D} type or the \mathcal{D}' type intervals, there are arbitrarily short intervals K of this type such that $z \in K$ and $S(K) \geq \tilde{\gamma}/12$. By Claim 5.7, for *both* types of intervals, there are arbitrarily short intervals K of this type such that $z \in K$ and $S(K) \leq 6\tilde{\beta}$. Since $\tilde{\gamma}/\tilde{\beta} > 72$, this means that one of the martingales $M^f, M^f_{1/3}$ does not converge on z . \square

5.2. Denjoy alternative, even for partial functions. Our version of the Denjoy alternative for a function f defined on the unit interval says that

$$(3) \quad \text{either } f'(z) \text{ exists, or } \overline{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty.$$

It is a consequence of the classical Denjoy (1907) -Young- (1912) - Saks (1937) theorem that for *any* function defined on the unit interval, the DA holds at almost every z . The actual result is in terms of right and left upper and lower Dini derivatives denoted $D^+f(z)$ (right upper) etc. Denjoy proved it for continuous, Young for measurable and Saks for all functions.

The result is used for instance to show that f' is always Borel (as a partial function). A paper by Alberti, Csornyei, Laczkovich, and Preiss (Real Anal. Exchange Vol. 26(1), 2000/2001, pp. 485-488) revisits the DA.

Definition 5.8. *A computable real z is given by a computable Cauchy name, i.e., a sequence $(q_n)_{n \in \mathbb{N}}$ of rationals converging to z such that $|q_{n+1} - q_n| \leq 2^{-n-1}$. Then $|z - q_n| \leq 2^{-n}$. If the Cauchy name is understood we sometimes write $(z)_n$ for q_n .*

Recall the following.

Definition 5.9. *A function g defined on the computable reals is called Markov computable if from a computable Cauchy name for x one can compute a computable Cauchy name for $g(x)$.*

Most relevant work of Demuth on the Denjoy alternative for effective functions is in this <http://dl.dropbox.com/u/370127/DemuthPapers/Demuth88PreprintDenjoySets.pdf>.

simply called *Demuth preprint* below. This later turned into the paper <http://dl.dropbox.com/u/370127/DemuthPapers/Demuth88PaperDenjoySets.pdf> Remarks on Denjoy sets; unfortunately a lot of things from the preprint are missing there.

Since the functions aren't total any more, we have to introduce "pseudo-derivatives" at z , taking the limit of slopes close to z where the function is defined. This is presumably what Demuth did. Consider a function g defined on $I_{\mathbb{Q}}$, the rationals in $[0, 1]$. For $z \in [0, 1]$ let

$$\underline{D}g(z) = \liminf_{h \rightarrow 0^+} \{S_g(a, b) : a, b \in I_{\mathbb{Q}} \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

$$\widetilde{D}g(z) = \limsup_{h \rightarrow 0^+} \{S_g(a, b) : a, b \in I_{\mathbb{Q}} \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

Note that Markov computable functions are continuous on the computable reals, so it does not matter which computable dense set of computable reals we take in the definition of these pseudo-derivatives. For a total continuous function g , we have $\underline{D}g(z) = \underline{D}g(z)$ and $\widetilde{D}g(z) = \overline{D}g(z)$. See last section of [7].

Suppose more generally we have a function f with domain containing $I_{\mathbb{Q}}$ we say that the *Denjoy alternative* holds if

$$(4) \quad \text{either } \widetilde{D}f(z) = \underline{D}f(z) < \infty, \text{ or } \widetilde{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty.$$

This is equivalent to (3) if the function is total and continuous.

5.3. Denjoy randomness coincides with computable randomness.

Definition 5.10 (Demuth preprint, page 4). *A real $z \in [0, 1]$ is called Denjoy random (or a Denjoy set) if for no Markov computable function g we have $\underline{D}g(z) = \infty$.*

In the Demuth preprint, page 6, it is shown that if $z \in [0, 1]$ is Denjoy random, then for every computable $f: [0, 1] \rightarrow \mathbb{R}$ the Denjoy alternative (3) holds at z . Combining this with the results in [7] we can now figure out what Denjoy randomness is, and also obtain a pleasing new characterization of computable randomness through differentiability of computable functions.

Theorem 5.11 (Demuth, Miller, Nies, Kučera). *The following are equivalent for a real $z \in [0, 1]$*

- (i) z is Denjoy random.
- (ii) z is computably random
- (iii) for every computable $f: [0, 1] \rightarrow \mathbb{R}$ the Denjoy alternative (3) holds at z .

Proof. (i) \rightarrow (iii) is the result of Demuth.

(iii) \rightarrow (ii) Let f be a nondecreasing computable function. Then f satisfies the Denjoy alternative at z . Since $\underline{D}f(z) \geq 0$, this means that $f'(z)$ exists.

This implies that z is computably random by [7, Thm. 4.1].

(ii) \rightarrow (i). Given a binary string σ , we will denote the open interval $(0.\sigma, 0.\sigma + 2^{-|\sigma|})$ by (σ) . We also write $S_f(\sigma)$ to mean $S_f(a, b)$ where $(a, b) = (\sigma)$.

By hypothesis z is incomputable, and in particular not a rational. Suppose that the function g is Markov computable and $\underline{D}g(z) = +\infty$. Choose dyadic rationals a, b such that $(a, b) = (\sigma)$ for some string σ , $z \in (a, b) \subseteq [0, 1]$ and $S_g(r, s) > 4$ for each r, s such that $z \in (r, s) \subseteq (a, b)$.

Define a computable martingale M on extensions $\tau \succeq \sigma$ that succeeds on (the binary expansion of) z . In the following τ ranges over such extensions.

Firstly, note that $S_g(\tau)$ is a computable real uniformly in τ . Furthermore, the function $\tau \rightarrow S_g(\tau)$ satisfies the martingale equality, and succeeds on z in the sense that its values are unbounded (even converge to ∞) along z . However, this function may have negative values; J. Miller has called this “betting with debt” because we can increase our capital at a string $\sigma 0$ beyond $2S_g(\sigma)$ by incurring a debt, i.e. negative value, at $S_g(\sigma 1)$. We now define a computable martingale M that succeeds on z and does not use betting with debt.

Let $M(\emptyset) = S_g(\emptyset)$. Suppose now that $M(\tau)$ has been defined and is positive.

Case 1. There is $u \in \{0, 1\}$ such that, where $v = 1 - u$, we have $S(\tau v)_1 < 1$ (this is the second term in the Cauchy name for the computable real $S(\tau v)$, which is at most $1/2$ away from that real). Then $S(\tau v) < 2$. By choice of a, b we now know that z is not an extension of τv . Thus, we let M double its capital along τu , let $M(\rho) = 0$ for all $\rho \succeq \tau v$. (The martingale M stops betting on these extensions.)

Case 2. Otherwise. Then $S(\tau v) > 0$ for $v = 0, 1$. We let M bet with the same betting factors as S_g :

$$M(\tau u) = M(\tau) \frac{S_g(\tau u)}{S_g(\tau)}$$

for $u = 0, 1$. Note that $M(\tau u) > 0$.

If Case 1 applies to infinitely many initial segments of the binary expansion of z , then M doubles its capital along z infinitely often. Since M has only positive values along z , this means that $\lim_n M(z \upharpoonright_n)$ fails to exist, whence z is not computably random by the effective version of the Doob martingale convergence theorem \square .

Otherwise, along z , M is eventually in Case 2. So M succeeds on z because S_g does. \square

Note that all we needed for the last implication was that $g(q)$ is a computable real uniformly in a rational $q \in I_{\mathbb{Q}}$. Thus, in Definition 5.10 we can replace Markov computability of g by this weaker hypothesis.

5.4. The Denjoy alternative for functions satisfying effectiveness notions weaker than computable.

This is work of Bienvenu, Hölzl and Nies who met at the LIAFA in Paris May-June.

Demuth [9] proved that the DA holds at what is now called Demuth random reals, for each Markov computable function. We show that in fact the much weaker notion of difference randomness is enough! Difference randomness was introduced by Franklin and Ng [11]. They showed that it is equivalent to being ML-random and Turing incomplete.

5.4.1. *Weak 2-randomness yields the DA for functions computable on $I_{\mathbb{Q}}$.* First we review some things from the last section of [7]. Let $I_{\mathbb{Q}} = [0, 1] \cap \mathbb{Q}$.

A function $f: \subseteq [0, 1] \rightarrow \mathbb{R}$ is called *computable on $I_{\mathbb{Q}}$* if $f(q)$ is defined and a computable real uniformly in the rational q .

For any rational p , let

$$\mathcal{C}(p) = \{z: \forall t > 0 \exists a, b [a \leq z \leq b \wedge 0 < b - a \leq t \wedge S_f(a, b) < p],$$

where t, a, b range over rationals. Since f is computable on $I_{\mathbb{Q}}$, the set

$$\{z: \exists a, b [a \leq z \leq b \wedge 0 < b - a \leq t \wedge S_f(a, b) < p]$$

is a Σ_1^0 set uniformly in t . Then $\mathcal{C}(p)$ is Π_2^0 uniformly in p . Furthermore,

$$(5) \quad \underline{D}f(z) < p \Rightarrow z \in \mathcal{C}(p) \Rightarrow \underline{D}f(z) \leq p.$$

Analogously we define

$$\tilde{\mathcal{C}}(q) = \{z: \forall t > 0 \exists a, b [a \leq z \leq b \wedge 0 < b - a \leq t \wedge S_f(a, b) > q].$$

Similar observations hold for these sets.

Theorem 5.12. *Let $f: \subseteq [0, 1] \rightarrow \mathbb{R}$ be computable on $I_{\mathbb{Q}}$. Then f satisfies the Denjoy alternative at every weakly 2-random real z .*

Proof. We adapt the classical proof in [5, p. 371] to the case of pseudo-derivatives. We analyse the arithmetical complexity of exception sets to conclude that weak 2-randomness is enough for the DA to hold.

We let a, b, p, q range over $I_{\mathbb{Q}}$. Recall Definition 5.8. For each $r < s$, $r, s \in I_{\mathbb{Q}}$, and for each $n \in \mathbb{N}$, let

$$(6) \quad E_{n,r,s} = \{x \in [0, 1]: \forall a, b [r \leq a \leq x \leq b \leq s \rightarrow S_f(a, b)_0 > -n + 1]\}.$$

Note that $E_{n,r,s}$ is a Π_1^0 class. For every n we have the implications

$$\underline{D}f(z) > -n + 2 \rightarrow \exists r, s [z \in E_{n,r,s}] \rightarrow \underline{D}f(z) > -n.$$

To show the DA (4) at z , we may assume that $\underline{D}f(z) > -\infty$ or $\tilde{D}f(z) < \infty$. If the second condition holds we replace f by $-f$, so we may assume the first condition holds. Then $z \in E_{n,r,s}$ for some r, s, n as above. Write $E = E_{n,r,s}$.

For $p < q$, the class $E \cap \mathcal{C}(p) \cap \tilde{\mathcal{C}}(q)$ is Π_2^0 . By (5) it suffices to show that each such class is null. For this, we show that for a.e. $x \in E$, we have $\underline{D}f(z) = \tilde{D}f(x)$. This remaining part of the argument is entirely within classical analysis. Replacing f by $f(x) + nx$, we may assume that for $x \in E$, we have

$$\forall a, b [r \leq a \leq x \leq b \leq s \rightarrow S_f(a, b)_0 > 1].$$

Let $f_*(x) = \sup_{a \leq x} f(a)$. Then f_* is nondecreasing on E . Let g be an arbitrary nondecreasing function defined on $[0, 1]$ that extends f_* . Then by a classic theorem of Lebesgue, $L(x) := g'(x)$ exists for a.e. $x \in [0, 1]$.

The following is a definition from classic analysis due to E.P. Dolzhenko, 1967 (see for instance [5, 5.8.124] but note the typo there).

Definition 5.13. *We say that E is porous at x via $\epsilon > 0$ if for each $\alpha > 0$ there exists β with $0 < \beta \leq \alpha$ such that $(x - \beta, x + \beta)$ contains an open interval of length $\epsilon\beta$ that is disjoint from E . We say that E is porous at x if it is porous at x via some ϵ .*

By the Lebesgue density theorem, the points in E at which E is porous form a null set.

Claim 5.14. *For each $x \in E$ such that $L(x)$ is defined and E is not porous at x , we have $\tilde{D}f(x) \leq L(x) \leq \underline{D}f(x)$.*

Since $\underline{D}f(x) \leq \tilde{D}f(x)$, this establishes the theorem.

To prove the claim, we show $\tilde{D}f(x) \leq L(x)$, the other inequality being symmetric. Fix $\epsilon > 0$. Choose $\alpha > 0$ such that

$$(7) \quad \forall u, v \in E [(u \leq x \leq v \wedge 0 < v - u \leq \alpha) \rightarrow S_{f_*}(u, v) \leq L(x)(1 + \epsilon)];$$

furthermore, since E is not porous at x , for each $\beta \leq \alpha$, the interval $(x - \beta, x + \beta)$ contains no open subinterval of length $\epsilon\beta$ that is disjoint from E . Now suppose that $a, b \in I_{\mathbb{Q}}$, $a < x < b$ and $\beta = 2(b - a) \leq \alpha$. There are $u, v \in E$ such that $0 \leq a - u \leq \epsilon\beta$ and $0 \leq v - b \leq \epsilon\beta$. Since $u, v \in E$ we have $f_*(u) \leq f(a)$ and $f(b) \leq f_*(v)$. Therefore $v - u \leq b - a + 2\epsilon\beta = (b - a)(1 + 4\epsilon)$. It follows that

$$S_f(a, b) \leq \frac{f_*(v) - f_*(u)}{b - a} \leq S_{f_*}(u, v)(1 + 4\epsilon) \leq L(x)(1 + 4\epsilon)(1 + \epsilon). \quad \square$$

5.4.2. Difference randomness yields the Denjoy alternative for Markov computable functions. We slightly reformulate the definition of difference randomness by Franklin and Ng [11].

Definition 5.15. A *difference test* is given by a Π_1^0 class $P \subseteq [0, 1]$, together with a uniformly Σ_1^0 sequence of classes $(U_n)_{n \in \mathbb{N}}$ where $U_n \subseteq [0, 1]$, such that $\lambda(P \cap U_n) \leq 2^{-n}$ for each n . A real z fails the test if $z \in P \cap \bigcap_n U_n$, otherwise z passes the test. We say z is difference random if it passes each difference test.

Franklin and Ng [11] show that

$$z \text{ difference random} \Leftrightarrow z \text{ is ML-random} \wedge \text{Turing incomplete.}$$

The following direct proof that no left-c.e. real α (such as Chaitin's Ω) is difference random might be helpful to understand the concept of difference tests. Let $P = [\alpha, 1]$. Let $U_n = [0, (i + 1)2^{-n}]$ where $i \in \mathbb{N}$ is largest such that $i2^{-n} < \alpha$. Then $P, (U_n)_{n \in \mathbb{N}}$ is a difference test and $\alpha \in P \cap \bigcap_n U_n$.

Theorem 5.16. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be Markov computable. Then f satisfies the Denjoy alternative at every difference random real z .*

Proof. Note that each Markov computable function is computable on $I_{\mathbb{Q}}$. We will show that under the stronger condition of Markov computability, the relevant null sets in the proof of the foregoing Theorem 5.12 are effective null sets in the sense of difference randomness. The proof relies on Lemmas 5.18 and 5.19, which will be established in Subsection 5.4.3 below.

Given $n \in \mathbb{N}$ and $r < s$ in $I_{\mathbb{Q}}$, define the set E as above. As before, we may assume that for $x \in E$, we have $\forall a, b [r \leq a \leq x \leq b \leq s \rightarrow S_f(a, b)_0 > 1]$, and hence the function $f_*(x) = \sup_{a \leq x} f(a)$ is nondecreasing on E .

Firstly, we show that some total nondecreasing extension g of $h = f_* \upharpoonright_E$ can be chosen to be computable. Since f is Markov computable, we obtain the following.

Claim 5.17. *The function $f_* \upharpoonright_E$ is computable.*

To see this, recall that p, q range over $I_{\mathbb{Q}}$, and let $f^*(x) = \inf_{q \geq x} f(q)$. If $x \in E$ and $f_*(x) < f^*(x)$ then x is computable: fix a rational d in between these two values. Then $p < x \leftrightarrow f(p) < d$, and $q > x \leftrightarrow f(q) > d$. Hence x is both left-c.e. and right-c.e., and therefore computable. Now a Markov computable function is continuous at every computable x . Thus $f_*(x) = f^*(x)$ for each x in E .

To compute $f(x)$ for $x \in E$ up to precision 2^{-n} , we can now simply search for rationals $p < x < q$ such that $0 < f(q)_{n+2} - f(p)_{n+2} < 2^{-n-1}$, and output $f(p)_{n+2}$. If during this search we detect that $x \notin E$, we stop. This shows the claim.

Lemma 5.18. *Let $h: \subseteq [0, 1] \rightarrow \mathbb{R}_0^+$ be a computable function that is defined and non-decreasing on a Π_1^0 class E . Then $h \upharpoonright_E$ can be extended to a function $g: [0, 1] \rightarrow \mathbb{R}_0^+$ that is computable and non-decreasing on $[0, 1]$.*

By [7, Thm. 5.1] we know that $L(x) := g'(x)$ exists for each computably random (and hence certainly each ML-random) real x .

To show that in fact $\underline{D}f(z) = \tilde{D}f(z) = L(z)$ for each difference random real z , we need the following.

Lemma 5.19. *Let $E \subseteq [0, 1]$ be a Π_1^0 class. If $z \in E$ is difference random, then E is not porous at z .*

Given the lemma, we can conclude the proof of the theorem by invoking Claim 5.14. \square

5.4.3. Proving the two lemmas.

Lemma 5.20. *Let E be a nonempty Π_1^0 class. Let $h: \subseteq [0, 1] \rightarrow \mathbb{R}_0^+$ be a computable function with domain containing E . Then $\sup_{x \in E} \{h(x)\}$ is right-c.e. and $\inf_{x \in E} \{h(x)\}$ is left-c.e. uniformly in an index for E .*

Proof. We proof the statement for the supremum; the proof for the infimum is analogous. We use the signed digit representation of reals, that is every real is represented by an infinite sequence in $\{-1, 0, 1\}^\infty$.

We run in parallel an enumeration of E^c and for all $x \in [0, 1]$ (given by a Cauchy name $(x_n)_{n \in \mathbb{N}}$) the computations of $h(x)$ up to precision 2^{-n} . That is, we want to compute $(h(x))_n$, the n -th entry of a Cauchy name for $h(x)$.

Due to uniform continuity, for each n there is a number n' such that for all x in order to accomplish the computation it suffices to have access to the initial segment $(x_0, \dots, x_{n'})$ of the Cauchy name of x . When the computation of $(h(x))_n$ halts for some x it also halts for all other x' which have a Cauchy name that begins with $(x_0, \dots, x_{n'})$, since the computation is clearly the same. We do not know n' , so we build a tree of computations that branches into three directions ($-1, 0$ and 1) whenever we access a new entry of the Cauchy name of the input. We remove a branch of the tree when it gets covered by E^c . Due to the existence of n' the tree will remain finite.

Write $\sup_{x \in E} \{h(x)\}[n]$ for the approximation to the value $\sup_{x \in E} \{h(x)\}$ that we achieve when we proceed as described with precision level 2^{-n} . If we

increase the precision, more of E^c may get enumerated before halting has occurred everywhere on the tree; so we see that the sequence $(\sup_{x \in E} \{h(x)\}[n] + 2^{-n})_{n \rightarrow \infty}$ is a right-c.e. approximation to $\sup_{x \in E} \{h(x)\}$. \square

Proof of Lemma 5.18. Since E is compact and closed, h is uniformly continuous on E , that is, for every $\varepsilon > 0$ there exists a *single* $\delta(\varepsilon) > 0$ such that for any point $x \in E$ the continuity condition

$$|y - x| < \delta(\varepsilon) \Rightarrow |h(y) - h(x)| < \varepsilon$$

is satisfied.

Idea. We do not know $\delta(\varepsilon)$ for a given ε , but we can search for it using in parallel the following construction for different candidate δ 's:

We split the unit interval into intervals $(I_k)_k$ of length δ and write l_k for the left border point of I_k . Write i_k for $\inf\{h(x) \mid x \in I_k \cap E\}$ and s_k for $\sup\{h(x) \mid x \in I_k \cap E\}$ if these values exist. We use lemma 5.20 to approximate i_k and s_k for all intervals, and at the same time we enumerate E^c .

We do this until we have found a δ (called ε -fit) such that every interval has been dealt with; by this we mean that for every interval I_k we have either covered I_k with E^c , or we have found that $s_k - i_k < \varepsilon$, that is we already know h up to precision ε . In the latter case we set our approximation to h to be the line from point (l_k, i_k) to point (l_{k+1}, s_k) ; on the remaining intervals (the former case) we interpolate linearly. Call the new function g_0 . We can then output $g_0(x)$ up to precision ε at any point $x \in [0, 1]$.

A problem with this construction. The following problem can occur with g_0 : Assume we have for some ε found a δ that is ε -fit. We construct g_0 as described and interpolate linearly on, say, the maximal connected sequence I_k, \dots, I_{k+n} , all contained in E^c . But if we look at the same construction for g_0 at a better precision $\varepsilon' < \varepsilon$, we might actually enumerate more of E^c until we find an ε' -fit δ' , and this might extend the sequence I_k, \dots, I_{k+n} to, say, $I_{k-i}, \dots, I_{k+n+j}$, all contained in E^c . The linear interpolation on this sequence of intervals would then be *significantly flatter* than at level ε . So for some $x \in I_{k-i} \cup \dots \cup I_{k+n+j}$ we might have that the approximation to g_0 with precision ε differs by more than ε from the approximation to g_0 with precision ε' , which is not allowed.

To fix this problem we need to define g inductively over all precision levels, and “commit” to all linear interpolations that have happened at earlier precision levels, as will be described now.

Formal construction. Assume we want to compute the n -th entry of a Cauchy name for $g(x)$, that is we want to compute $g(x)$ up to precision 2^{-n} . We say that *we are at precision level n* . We do not know $\delta(2^{-n})$ so we do the following with $\delta = 2^{-p}$ in parallel for all p until we find a δ that is n -fit, defined as follows:

Split the interval $[0, 1]$ into intervals of length δ and write

$$I_k = [(k-1) \cdot 2^{-p}, k \cdot 2^{-p})$$

for the k -th interval and $l_k = (k-1) \cdot 2^{-p}$ for the left border point of I_k . For mathematical precision set $I_p := [1 - 2^{-p}, 1]$. Call an interval I_k n -treated if there exists a smaller precision level $n' < n$ where I_k

has been covered by E^c and therefore a linear interpolation on I_k has been defined. We say that δ is *n-fit* if

- for every interval I_k we have that
 - (1) I_k is n -treated or
 - (2) $I_k \cap E = \emptyset$ or
 - (3) $s_k - i_k < 2^{-n}$
- and if for all k , where both I_k and I_{k+1} fulfill condition 3, we have $i_{k+1} < s_k$; that is, we have that intervals that directly follow each other have a “vertical overlap” in their approximations.

The following linear interpolation is an 2^{-n} -close approximation to g :

First, inductively replay the construction for all precision levels $n' < n$ to find all n -treated intervals. For the remaining intervals, run in parallel the right-c.e. and left-c.e. approximations to s_k and i_k , respectively, and the enumeration of E^c , until for every interval either condition 2 or 3 are satisfied.

Build the following piecewise linear function: For all intervals that are already n -treated, keep the linear interpolations from the earlier precision level $n - 1$. In all remaining intervals I_k that fulfill condition 3 we set g to be the line from point (l_k, i_k) to (l_{k+1}, s_k) , that is, for $x \in I_k$ we let

$$g(x) = i_k + (x - l_k) \cdot \frac{(s_k - i_k)}{l_{k+1} - l_k} = i_k + (x - l_k) \frac{(s_k - i_k)}{\delta}.$$

Now look at the remaining intervals that have not yet been assigned a linear interpolation. In every maximal connected sequence I_k, \dots, I_{k+n} of such intervals every interval must fulfill condition 2. We interpolate linearly over I_k, \dots, I_{k+n} in the straightforward way, that is we draw a line from point $(s_k, g(l_k))$ to point $(s_{k+n+1}, g(l_{k+n+1}))$ (strictly speaking $g(l_k)$ is not yet defined, so use $\lim_{x \rightarrow l_k} g(x)$ instead).

Verification. It is clear that g is everywhere defined and non-decreasing. Whenever h was defined on a point $x \in I_k$ inside E , g gets assigned a value between i_k and s_k and since $s_k > h(x) > i_k$ and $s_k - i_k < 2^{-n}$ we have $|h(x) - g(x)| < 2^{-n}$. To see that g is computable note that s_k and i_k are defined on any interval that is not entirely contained in E^c and that these values can be approximated in a right-c.e. and left-c.e. way, respectively, by using lemma 5.20. \square

Proof of Lemma 5.19. In this proof, we say that a string σ meets \mathcal{C} if $\llbracket \sigma \rrbracket \cap \mathcal{C} \neq \emptyset$.

Fix $c \in \mathbb{N}$ such that \mathcal{C} is porous at z via 2^{-c+2} . For each string σ consider the set of minimal “porous” extensions at stage t ,

$$N_t(\sigma) = \left\{ \rho \succeq \sigma \mid \exists \tau \succeq \sigma \left[\begin{array}{l} |\tau| = |\rho| \wedge |0.\tau - 0.\rho| \leq 2^{-|\tau|+c} \wedge \\ \llbracket \tau \rrbracket \cap \mathcal{C}_t = \emptyset \wedge \rho \text{ is minimal with this property} \end{array} \right] \right\}.$$

We claim that

$$(8) \quad \sum_{\substack{\rho \in N_t(\sigma) \\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} \leq (1 - 2^{-c-2}) 2^{-|\sigma|}.$$

To see this, let R be the set of strings ρ in (8). Let V be the set of prefix-minimal strings that occur as witnesses τ in (8). Then the open sets generated by R and by V are disjoint. Thus, if r and v denote their measures, respectively, we have $r + v \leq 2^{-|\sigma|}$. By definition of $N_t(\sigma)$, for each $\rho \in R$ there is $\tau \in V$ such that $|0.\tau - 0.\rho| \leq 2^{-|\tau|+c}$. This implies $r \leq 2^{c+1}v$. The two inequalities together imply (8) because $r \leq 2^{c+1}(1-r)$ implies $r \leq 1 - 1/(2^c + 1) + 1$.

Note that by the formal details of this definition even the “holes” τ are ρ 's, and therefore contained in the sets $N_t(\sigma)$. This will be essential for the proof of the first of the following two claims. At each stage t of the construction we define recursively a sequence of anti-chains as follows.

$$B_{0,t} = \{\emptyset\}, \text{ and for } n > 0: B_{n,t} = \bigcup \{N_t(\sigma) : \sigma \in B_{n-1,t}\}$$

Claim. If a string ρ is in $B_{n,t}$ then it has a prefix ρ' in $B_{n,t+1}$.

This is clear for $n = 0$. Suppose inductively that it holds for $n - 1$. Suppose further that ρ is in $B_{n,t}$ via a string $\sigma \in B_{n-1,t}$. By the inductive hypothesis there is $\sigma' \in B_{n-1,t+1}$ such that $\sigma' \preceq \sigma$. Since $\rho \in N_t(\sigma)$, ρ is a viable extension of σ' at stage $t + 1$ in the definition of $N_{t+1}(\sigma')$, except maybe for the minimality. Thus there is $\rho' \preceq \rho$ in $N_{t+1}(\sigma')$. \diamond

Claim. For each n, t , we have $\sum \{2^{-|\rho|} : \rho \in B_{n,t} \wedge \rho \text{ meets } \mathcal{C}\} \leq (1 - 2^{-c-2})^n$. This is again clear for $n = 0$. Suppose inductively it holds for $n - 1$. Then, by (8),

$$\sum_{\substack{\rho \in B_{n,t} \\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} = \sum_{\substack{\sigma \in B_{n-1,t} \\ \sigma \text{ meets } \mathcal{C}}} \sum_{\substack{\rho \in N_t(\sigma) \\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} \leq \sum_{\substack{\sigma \in B_{n-1,t} \\ \sigma \text{ meets } \mathcal{C}}} 2^{-|\sigma|}(1 - 2^{-c-2}) \leq (1 - 2^{-c-2})^n.$$

This establishes the claim. \diamond

Now let $U_n = \bigcup_t \llbracket B_{n,t} \rrbracket$. Clearly the sequence $(U_n)_{n \in \mathbb{N}}$ is uniformly effectively open. By the first claim, $U_n = \bigcup_t \llbracket B_{n,t} \rrbracket$ is a nested union, so the second claim implies that $\lambda(\mathcal{C} \cap U_n) \leq (1 - 2^{-c-2})^n$. We show $z \in \bigcap U_n$ by induction on n . Clearly $z \in U_0$. If $n > 0$ suppose inductively $\sigma \prec z$ where $\sigma \in \bigcup_t B_{n-1,t}$. Since z is random there is η such that $\sigma \prec \eta \prec z$ and η ends in $0^c 1^c$. Every interval $(a, b) \subseteq [0, 1]$ contains an interval of the form $\llbracket \rho \rrbracket$ for a dyadic string ρ such that the length of $\llbracket \rho \rrbracket$ is no less than $(b - a)/4$. Thus, since \mathcal{C} is porous at z via 2^{-c+2} , there is t , $\rho \succeq \eta$ and τ satisfying the condition in the definition of $N_t(\sigma)$. By the choice of η one verifies that $\tau \succeq \sigma$. Thus $z \in U_n$.

Now take a computable subsequence $(U_{g(n)})_{n \in \mathbb{N}}$ such that $\lambda(\mathcal{C} \cap U_{g(n)}) \leq 2^{-n}$ to obtain a difference test that z fails. \square

5.5. Sets of non-differentiability for single computable functions
 $f: [0, 1] \rightarrow \mathbb{R}$. The plan is to characterize the sets $\{z: f'(z) \text{ is undefined}\}$ for computable functions on the unit interval. We also want to consider the case when the functions have additional analytical properties, such as being of bounded variation, or being Lipschitz.

5.5.1. *The classical case.* Zahorski [27] showed the following (also see Fowler and Preiss [10]).

Theorem 5.21. *The sets of non-differentiability of a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ are exactly of the form $L \cup M$, where L is G_δ (ie boldface Π_2^0) and M is a $G_{\delta\sigma}$ (ie boldface Σ_3^0) null set.*

Proof. Given f , the set

$$L = \{z: \overline{D}f(z) = \infty \wedge \underline{D}f(z) = -\infty\}$$

is G_δ . Let

$$M = \{z: z \notin L \wedge \exists p < q [p, q \in \mathbb{Q} \wedge \underline{D}f(z) < p \wedge q < \overline{D}f(z)]\}.$$

Then M is $G_{\delta\sigma}$.

Note that $f'(z)$ is undefined iff $z \in L \cup M$, and M is null by the Denjoy alternative.

For the converse direction, we are given L, M and have to build f . We may assume that L, M are disjoint, else replace M by $M - L$. Zahorski builds a continuous function g which is non-differentiable exactly in L (this is hard). Fowler and Preiss build a Lipschitz function h nondifferentiable exactly in M . Now let $f = g + h$.

5.5.2. *The algorithmic case.* If f is computable, then the set L defined above is lightface Π_2^0 , and M is lightface Σ_3^0 .

For the other direction, it's not clear what happens now. Suppose L is lightface Π_2^0 and M is lightface Σ_3^0 . Fowler and Preiss don't say what Zahorski did, and the original paper is probably awful to read. However, the proof [7, Thm 3.1] is likely to come close. There, a Π_2^0 set L is given, and one builds a computable function g that is nondifferentiable in L and differentiable outside a Π_2^0 set $\widehat{L} \supseteq L$, where after modifying that construction a bit, one can ensure that $\widehat{L} - L$ is null.

Since the Fowler-Preiss function for M is Lipschitz, we can't hope this part of the construction works algorithmically, because a computable Lipschitz function is differentiable at all computably randoms by [7, Section 5]. \square

5.6. Extending the results of [7] to higher dimensions. So far, most of the interaction between randomness and computable analysis is restricted to the 1-dimensional case. Anyone who has done calculus will confirm that differentiation becomes more interesting (and challenging) in higher dimensions. When randomness notions are defined for infinite sequences of bits, that is points in Cantor space $2^\mathbb{N}$, it is irrelevant to proceed to higher dimensions, because $(2^\mathbb{N})^n$ is in all aspects (metric, measure) equivalent to $2^\mathbb{N}$. This is clearly no longer true when randomness for n -tuples of reals (yes, reals) in the unit interval $[0, 1]$ is studied, because the n -cube $[0, 1]^n$ for $n > 1$ is much more complex than the unit interval.

What I am trying to say is that the identification of Cantor space and $[0, 1]$ doesn't always make sense any longer for higher dimensions, because methods typical for n -space aren't there in the setting of $(2^\mathbb{N})^n \cong 2^\mathbb{N}$. It makes sense in the setting of L_p computability, though. Also we can still use the identification to *define* notions such as computable randomness in

$[0, 1]^n$, if pure measure theory won't do it (it does for Martin-Löf and Schnorr randomness).

Currently several researchers investigate extensions of the results in [7] to higher dimensions. Already Pathak [24] showed that a weak form of the Lebesgue differentiation theorem holds for Martin-Löf random points in the n -cube $[0, 1]^n$. Work in progress of Rute, and Pathak, Simpson, and Rojas might strengthen this to Schnorr random points in the n -cube. On the other hand, functions of bounded variation can be defined in higher dimension [5, p. 378], and one might try to characterize Martin-Löf randomness in higher dimensions via their differentiability.

By work submitted May 2012 of [12], computable randomness can be characterized via differentiability of computable Lipschitz functions. The obvious analog of computable randomness in higher dimensions has been introduced in [7, 25].

Definition 5.22. *Call a point $x = (x_1, \dots, x_n)$ in the n -cube computably random if no computable martingale succeeds on the binary expansions of x_1, \dots, x_n joined in the canonical way (alternating between the sequences).*

Rute studies it via measures. Differentiability of Lipschitz functions in higher dimensions has been studied to great depth (see for instance [1]). Effective aspects of this theory could be used as an approach to such a randomness notion. One could investigate whether this is equivalent to differentiability at x of all computable Lipschitz functions. In [12] it is shown that if $z \in [0, 1]^n$ is not computably random, some computable Lipschitz function is not differentiable at z (in fact some partial fails to exist). This is done by a straightforward reduction to the 1-dimensional case.

In any computable probability space, to be weakly 2-random means to be in no null Π_2^0 class. For weak 2-randomness in n -cube, new work of Nies and Turetsky yields the analog of [7, Thm 3.1]. The following writeup is due to Turetsky.

Theorem 5.23 (almost). *Let $z \in [0, 1]^n$. Then the following are equivalent:*

- (1) *z is weakly 2-random;*
- (2) *each a.e. differentiable computable function $f: [0, 1]^n \rightarrow \mathbb{R}$ is differentiable at z ;*
- (3) *each a.e. differentiable computable function $f: [0, 1]^n \rightarrow \mathbb{R}$ is Gâteaux differentiable at z ; and*
- (4) *each a.e. differentiable computable function $f: [0, 1]^n \rightarrow \mathbb{R}$ has all partial derivatives at z .*

Proof. (2) \Rightarrow (3) \Rightarrow (4) are facts from classical analysis. We show (1) \Rightarrow (4), (1) + (4) \Rightarrow (2) and (4) \Rightarrow (1).

(1) \Rightarrow (4). Suppose z is weakly 2-random and f is an a.e. differentiable computable function. Fix coordinate i . For $\vec{a} \in [0, 1]^n$, $h \in \mathbb{R}$, let

$$S_f^i(\vec{a}, h) = \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{h}.$$

Recall that $\partial f / \partial x_i$ exists at \vec{a} precisely if the upper derivative $\overline{D}^i f(\vec{a}) = \limsup_{|h| \rightarrow 0} S_f^i(\vec{a}, h)$ and the lower derivative $\underline{D}^i f(\vec{a}) = \liminf_{|h| \rightarrow 0} S_f^i(\vec{a}, h)$ are finite and equal.

For q a rational, $\overline{D}^i f(\vec{a}) \geq q$ is equivalent to the formula.

$$(\forall p < q)(\forall \delta > 0)(\exists |h| < \delta)[S_f^i(\vec{a}, h) > p]$$

By density, we can take p and δ to range over the rationals. Since f is computable, and thus continuous, we can take h to range over the rationals. Thus $\{\vec{a} \mid \overline{D}^i f(\vec{a}) \geq q\}$ is a Π_2^0 -set uniformly in q . Symmetrically, so is $\{\vec{a} \mid \underline{D}^i f(\vec{a}) \leq q\}$. Then the \vec{a} such that $\partial f / \partial x_i$ does not exist are precisely those \vec{a} satisfying

$$(\forall q)[\overline{D}^i f(\vec{a}) \geq q] \vee (\forall q)[\underline{D}^i f(\vec{a}) \leq q] \vee (\exists q, p)[\underline{D}^i f(\vec{a}) \leq q < p \leq \overline{D}^i f(\vec{a})].$$

This is a Σ_3^0 -set contained in the set of all points at which f is not differentiable, and thus has measure 0. Thus it cannot contain z , and so $\partial f / \partial x_i$ exists at z .

(1) + (4) \Rightarrow (2). Suppose z is weakly 2-random, f is an a.e. differentiable computable function and $\frac{\partial f}{\partial x_i}(z)$ exists for every i . Let $S_f^i(\vec{a}, h)$ be as defined above.

For $\vec{a} \in [0, 1]^n$ and $h \in \mathbb{R}$, let

$$J_f(\vec{a}, h) = [S_f^1(\vec{a}, h) \quad \dots \quad S_f^n(\vec{a}, h)].$$

By definition, $\lim_{|h| \rightarrow 0} J_f(\vec{a}, h) = J_f(\vec{a})$, the Jacobian of f at \vec{a} (when this exists).

Again by definition, the derivative of f exists at \vec{a} if $J_f(\vec{a})$ exists and

$$\lim_{\|\vec{h}\| \rightarrow 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h}}{\|\vec{h}\|} = 0.$$

By continuity of f , we can take \vec{h} to have rational coordinates.

Let X be the Π_3^0 -set consisting of those a satisfying

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall \|\vec{h}\| < \delta)(\forall |b| < \delta) \frac{|f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a}, b)\vec{h}|}{\|\vec{h}\|} < \epsilon.$$

Here ϵ, δ and b are rationals, and \vec{h} has rational coordinates. We argue first that X contains every point at which f is differentiable.

Suppose \vec{a} is a point at which f is differentiable. Fix $\epsilon > 0$. Let δ be sufficiently small that

$$(\forall \|\vec{h}\| < \delta) \frac{|f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h}|}{\|\vec{h}\|} < \epsilon/2,$$

and also

$$(\forall |b| < \delta) \|(J_f(\vec{a}, b) - J_f(\vec{a}))^T\| < \epsilon/2.$$

Here we treat $J_f(\vec{a}, b) - J_f(\vec{a})$ as a row vector. Then for any $\|\vec{h}\| < \delta$ and $|b| < \delta$,

$$\begin{aligned} \frac{\left| f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a}, b)\vec{h} \right|}{\|\vec{h}\|} &= \frac{\left| f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h} + J_f(\vec{a})\vec{h} - J_f(\vec{a}, b)\vec{h} \right|}{\|\vec{h}\|} \\ &\leq \frac{\left| f(\vec{a} + \vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h} \right|}{\|\vec{h}\|} + \frac{\left| (J_f(\vec{a}) - J_f(\vec{a}, b))\vec{h} \right|}{\|\vec{h}\|} \\ &< \epsilon/2 + \frac{\|(J_f(\vec{a}) - J_f(\vec{a}, b))^T\| \|\vec{h}\|}{\|\vec{h}\|} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus X contains every point at which f is differentiable, and so X has full measure. So $z \in X$.

Next we show that f is differentiable at z . Fix $\epsilon > 0$. Let δ be sufficiently small that

$$(\forall \|\vec{h}\| < \delta)(\forall |b| < \delta) \frac{\left| f(z + \vec{h}) - f(z) - J_f(z, b)\vec{h} \right|}{\|\vec{h}\|} < \epsilon/2,$$

and fix some $|b| < \delta$ such that

$$\|(J_f(z, b) - J_f(z))^T\| < \epsilon/2.$$

Then, similar to the above, for any $\|\vec{h}\| < \delta$,

$$\begin{aligned} \frac{\left| f(z + \vec{h}) - f(z) - J_f(z)\vec{h} \right|}{\|\vec{h}\|} &= \frac{\left| f(z + \vec{h}) - f(z) - J_f(z, b)\vec{h} + J_f(z, b)\vec{h} - J_f(z)\vec{h} \right|}{\|\vec{h}\|} \\ &\leq \frac{\left| f(z + \vec{h}) - f(z) - J_f(z, b)\vec{h} \right|}{\|\vec{h}\|} + \frac{\left| (J_f(z, b) - J_f(z))\vec{h} \right|}{\|\vec{h}\|} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus f is differentiable at z .

(4) \Rightarrow (1). □

5.7. The Lebesgue differentiation theorem. A version of the Lebesgue differentiation theorem in one dimension can be found, for instance, in [5, Section 5.4].

Theorem 5.24. *Let $g \in L_1([0, 1])$. Then λ -almost surely,*

$$g(z) = \lim_{r \rightarrow 0} \frac{1}{r} \int_z^{z+r} g \, d\lambda.$$

In other words, if $G(z) = \int_0^z g(x)dx$, then for λ -almost every z , $G'(z)$ exists and equals $g(z)$. (This explains the name given to the theorem.) In one dimension, the usual version becomes

$$g(z) = \lim_{r, s \rightarrow +0} \frac{1}{r+s} \int_{z-r}^{z+s} g \, d\lambda.$$

This is equivalent to the formulation above that $G'(z) = g(z)$ for a.e. z . For if f is a function on $[0, 1]$, then

$$f'(z) \text{ exists} \Leftrightarrow \lim_{r,s \rightarrow 0+} S_f(z-s, z+r) \text{ exists,}$$

in which case they are equal. To see this, note that if the limit on the right exists, it clearly equals $f'(z)$. Now suppose conversely that $c = f'(z)$ exists. Given $\varepsilon > 0$, choose $\delta > 0$ such that for $r, s > 0$,

$$\max(r, s) < \delta \text{ implies } |c - S_f(z-s, z)| \leq \varepsilon \wedge |c - S_f(z, z+r)| \leq \varepsilon.$$

Then

$$\begin{aligned} |(s+r)c - (f(z+r) - f(z-s))| &= |sc - sS_f(z-s, z) + rc - rS_f(z, z+r)| \\ &\leq |sc - sS_f(z-s, z)| + |rc - rS_f(z, z+r)| \\ &\leq \varepsilon(s+r) \end{aligned}$$

$$\text{Hence } |c - S_f(z-s, z+r)| \leq \varepsilon.$$

6. K -TRIVIALITY AND INCOMPRESSIBILITY IN COMPUTABLE METRIC SPACES

Nies and PhD student Melnikov (co-supervised with Khoussainov) worked in Auckland and on Rakiura/Stewart Island (December 2010).

For more detail than this, see their paper “<http://dl.dropbox.com/u/370127/papers/MelnikovNiesK>” [18], although arguments and some definitions are a bit different in that improved version. Also see Nies’ Talk “<http://dl.dropbox.com/u/370127/talks>” -triviality in computable metric spaces” at the Cape Town 2011 CCR. (Both are available on Nies’ web page.)

6.1. Background on computable metric spaces.

Definition 6.1. Let (M, d) be a Polish space, and let $(q_i)_{i \in \mathbb{N}}$ be a dense sequence in M without repetitions. We say that $\mathcal{M} = (M, d, (q_i)_{i \in \mathbb{N}})$ is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k . We say that $(q_i)_{i \in \mathbb{N}}$ is a *computable structure* on (M, d) , and refer to the elements of the sequence $(q_i)_{i \in \mathbb{N}}$ as the *special points*.

Definition 6.2. A sequence $(p_s)_{s \in \mathbb{N}}$ of special points is called a *Cauchy name* if $d(p_s, p_t) \leq 2^{-s}$ for each $s \in \mathbb{N}, t \geq s$. Since the metric space is complete, $x = \lim_s p_s$ exists; we say that $(p_s)_{s \in \mathbb{N}}$ is a Cauchy name for x . Note that $d(x, p_s) \leq 2^{-s}$ for each s .

Definition 6.3. We say that a point in a computable metric space is *computable* if it has a computable Cauchy name.

If a computable metric space $\mathcal{M} = (M, d, (q_i)_{i \in \mathbb{N}})$ is fixed in the background, we will use the letters p, q etc. for special points. We may identify the special point q_i with $i \in \mathbb{N}$. Thus, we may view a Cauchy name as a function $\alpha: \mathbb{N} \rightarrow \mathbb{N}$. We also write $\lim_n \alpha(n) = x$ meaning that $\lim_n q_{\alpha(n)} = x$.

Example 6.4. The following are computable metric spaces.

- (i) The unit interval $[0, 1]$ with the usual distance function and the computable structure given by some effective listing without repetition of the rationals in $[0, 1]$, i.e., by fixing a computable bijection τ between \mathbb{N} and $\mathbb{Q} \cap [0, 1]$, and letting $q_n = \tau(n)$.
- (ii) The Baire space $\mathbb{N}^{\mathbb{N}}$ consisting of the functions $f: \mathbb{N} \rightarrow \mathbb{N}$ with the usual ultrametric distance function

$$d(f, g) = \max\{2^{-n} : f(n) \neq g(n)\},$$

(where $\max \emptyset = 0$). The computable structure is given by fixing some effective listing without repetition of the functions that are eventually 0. (Note that such functions can be described by strings in $\mathbb{N}^{<\omega}$ that don’t end in 0.)

- (iii) Cantor space $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, with the inherited distance function and computable structure.

6.2. K -triviality. We will generalize the usual definition of K -triviality in Cantor space

$$\exists b \forall n K(x \upharpoonright_n) \leq K(0^n) + b$$

to points in general computable metric spaces.

We first provide some preliminary material. Thereafter, we introduce our main concept.

K-triviality for functions. Fix some effective encoding of tuples x over \mathbb{N} by binary strings, so that $K(x)$ is defined for any such tuple.

Definition 6.5. We say that a function $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ is *K-trivial* if

$$\exists b \forall n K(\alpha \upharpoonright_n) \leq K(0^n) + b.$$

Proposition 6.6. A function α is *K-trivial* iff its graph $\Gamma_\alpha \Leftarrow \{\langle n, \alpha(n) \rangle : n \in \mathbb{N}\}$ is *K-trivial* in the usual sense of sets.

Proof. One uses that *K-triviality* implies lowness for K (see [22, Section 5.4]). \square

Solovay functions.

Recall that a computable function $h: \mathbb{N} \rightarrow \mathbb{N}$ is called a *Solovay function* [2] if $\forall r [K(r) \leq^+ h(r)]$, and $\exists^\infty r [K(r) =^+ h(r)]$.

Solovay [26] constructed an example of such a function. The following simpler recent example is due to Merkle. We include the short proof for completeness' sake.

Fact 6.7. There is a Solovay function h .

Proof. Let \mathbb{U} denote the optimal prefix-free machine. Given $r = \langle \sigma, n, t \rangle$, if t is least such that $\mathbb{U}_t(\sigma) = n$, define $h(r) = |\sigma|$. Otherwise let $h(r) = r$.

We have $K(r) \leq^+ h(r)$ because there is a prefix-free machine M which on input σ outputs $r = \langle \sigma, \mathbb{U}(\sigma), t \rangle$ if t is least such that $\mathbb{U}_t(\sigma)$ halts. If σ is also a shortest string such that $\mathbb{U}(\sigma) = n$, then we have $h(r) = |\sigma| = K(N) \leq^+ K(r)$. \square

The following generalizes the corresponding fact for sets also due to Merkle.

Fact 6.8. Let $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\forall r K(\alpha \upharpoonright_r) \leq h(r) + b$. Then α is *K-trivial* via a constant $b + O(1)$.

Proof. Given n , let σ be a shortest \mathbb{U} -description of n , and let t be least such that $\mathbb{U}_t(\sigma) = n$. Let $r = \langle \sigma, n, t \rangle$. Then

$$K(\alpha \upharpoonright_n) \leq^+ K(\alpha \upharpoonright_r) \leq h(r) + b = |\sigma| + b = K(n) + b.$$

\square

6.3. K-trivial points in computable metric space. Our principal notion is *K-triviality* in computable metric spaces. Recall from Definition 6.3 that a point x in a computable metric space \mathcal{M} is called computable if it has a computable Cauchy name. We define *K-triviality* in a similar fashion.

Definition 6.9. We say that a point x in a computable metric space is *K-trivial* if it has a Cauchy name that is *K-trivial* as a function.

Proposition 6.10. Let \mathcal{M}, \mathcal{N} be computable metric spaces, and let the map $F: \mathcal{M} \rightarrow \mathcal{N}$ be computable. If x is *K-trivial* in \mathcal{M} , then $F(x)$ is *K-trivial* in \mathcal{N} .

Proof. Let α be a *K-trivial* Cauchy name for x . Since F is computable, there is a Cauchy name $\beta \leq_T \alpha$ for $F(x)$. Then β is *K-trivial* by Proposition 6.6 and the result of [21] that *K-triviality* for sets is closed downward under \leq_T . \square

If F is effectively uniformly continuous (say Lipschitz), then one can also give a direct proof which avoids the hard result from [21]. Moreover, from a *K-triviality* constant for x one can effectively obtain a *K-triviality* constant for $F(x)$, which is not true in the general case.

Since the identity map is Lipschitz, we obtain:

Corollary 6.11. *K-triviality* in a computable metric space is invariant under changing of the computable structure to an equivalent one. More specifically, if x is *K-trivial* for b with respect to a computable structure, then x is *K-trivial* for $b + O(1)$ with respect to an equivalent structure.

6.4. Existence of non-computable K -trivial points.

Example 6.12. *There is a computable metric space \mathcal{M} with a noncomputable point such that the only K -trivial points are the computable points.*

Proof. Let \mathcal{M} be the computable metric space with domain $\{\Omega_s : s \in \mathbb{N}\} \cup \{\Omega\}$, the metric inherited from the unit interval and with the computable structure given by $q_s = \Omega_s$.

Suppose g is a Cauchy name for Ω in \mathcal{M} . Then, for a rational p , we have $p > \Omega \leftrightarrow \exists n[g(n) + 2^{-n} < p]$. Thus $\{p \in \mathbb{Q} : p < \Omega\} \leq_T g$. This set is Turing complete, so g is not K -trivial. \square

A metric space is said to be *perfect* if it has no isolated points. In the following we take Cantor space $2^{\mathbb{N}}$ as the computable metric space with the usual computable structure of Example 6.4.

Proposition 6.13. [6, Prop. 6.2] *Suppose the computable metric space \mathcal{M} is perfect. Then there is a computable injective map $F : 2^{\mathbb{N}} \rightarrow \mathcal{M}$ which is Lipschitz with constant 1.*

Given $\delta > 0$, we denote by $B(x, \delta)$ the open ball of radius δ and with center in x , that is, $B(x, \delta) = \{y \in M : d(x, y) < \delta\}$.

Theorem 6.14. *Let \mathcal{M} be a computable perfect Polish space. Then for every special point $p \in M$ and every rational $\delta > 0$ there exists a non-computable K -trivial point $x \in B(p, \epsilon)$.*

Proof. Note that the closure in \mathcal{M} of $B(p, \delta)$ is again a perfect Polish space \mathcal{N} which has an inherited computable structure. By the result of Brattka and Gherardi there is a computable injective Lipschitz map $F : 2^{\mathbb{N}} \rightarrow \mathcal{N}$.

Let A be a non-computable K -trivial point in Cantor space. Then $x = F(A)$ is K -trivial in \mathcal{N} , and hence in \mathcal{M} , by Proposition 6.10; actually only the easier case of Lipschitz functions discussed after 6.10 is needed.

As Brattka and Gherardi point out before Prop. 6.2, the inverse of F is computable (on its domain). Thus, if x is computable then so is A , which is not the case. \square

6.5. An equivalent local definition of K -triviality. We fix a computable metric space $\mathcal{M} = (M, d, (q_n)_{n \in \mathbb{N}})$. When it comes to K -triviality for points in \mathcal{M} , one's first thought would be to directly adapt the definition of K -triviality of some point x in Cantor Space,

$$\exists b \forall n K(x \upharpoonright_n) \leq K(0^n) + b.$$

Recall from Example 6.4 that in Cantor space $2^{\mathbb{N}}$ we chose as the special points the sequences of bits that are eventually 0. Since we identify a special point $p = \alpha(n)$ with n , we get the following tentative definition:

$$(9) \quad \exists b \forall n \exists p [d(x, p) \leq 2^{-n} \wedge K(p) \leq K(n) + b].$$

This isn't right, though:

Proposition 6.15. *There is a Turing complete Π_1^0 set $A \in 2^{\mathbb{N}}$ satisfying condition (9)*

Proof. For a string α , let $g(\alpha)$ be the longest prefix of α that ends in 1, and $g(\alpha) = \emptyset$ if there is no such prefix. We say that a set A is *weakly K -trivial* if

$$\forall n [K(g(A \upharpoonright_n)) \leq^+ K(n)].$$

This is equivalent to (9). (Clearly, every K -trivial set is weakly K -trivial. Every c.e. weakly K -trivial set is already K -trivial.)

We now build a Turing complete Π_1^0 set A that is weakly K -trivial. We maintain the condition that

$$(10) \quad \forall i \forall w [\gamma_i < w \rightarrow K(w) > i],$$

where γ_i is the i -th element of A . This implies that A is Turing complete, as follows. We build a prefix-free machine N . When i enters \emptyset' at stage s , we declare that $N(0^i 1) = s$. This implies $K(s) \leq i + d$ for some fixed coding constant d . Now $i \in \emptyset' \leftrightarrow i \in \emptyset'_{\gamma_{i+d+1}}$, which implies $\emptyset' \leq_T A$.

We let $A = \bigcap A_s$, where A_s is a cofinite set effectively computed from s , $A_0 = \mathbb{N}$, $[s, \infty) \subseteq A_s$, and $A_{s+1} \subseteq A_s$ for each s . We view γ_i as a movable marker; γ_i^s denotes its position at stage s , which is the i -th element of A_s .

Construction of A and a prefix-free machine M .

Stage 0. Let $A_0 = \mathbb{N}$.

Stage $s > 0$. Suppose that there is w such that $i := K_s(w) < K_{s-1}(w)$. By convention, w is unique and $w < s$. Thus, there is a new computation $\mathbb{U}_s(\sigma) = w$ with $|\sigma| = i$ at stage s .

If $w \leq \gamma_i^{s-1}$ then let $A_s = A_{s-1}$. If $w > \gamma_i^{s-1}$ then, to maintain (10) at stage s , we move the marker γ_i : we let $A_s = A_{s-1} - [\gamma_i^{s-1}, s)$, which results in $\gamma_{i+k}^s = s + k$ for $k \geq i$, while $\gamma_j^s = \gamma_j^{s-1}$ for $j < i$.

In any case, declare $M(\sigma) = g(A_s \upharpoonright_w)$.

Verification. Clearly, each marker γ_i moves at most 2^{i+1} times, so $A = \bigcap_s A_s$ is an infinite co-c.e. set. Furthermore, condition (10) holds because it is maintained at each stage of the construction.

We show by induction on s that

$$(11) \quad \forall n [K(g(A_s \upharpoonright_n)) \leq^+ K_s(n)].$$

For $s = 0$ the condition is vacuous. Now suppose $s > 0$. We may suppose that w as in stage s of the construction exists, otherwise (11) holds at stage s by inductive hypothesis.

As in the construction let $i = K_s(w)$, and let σ be the string of length i such that $\mathbb{U}_s(\sigma) = w$. If $w \leq \gamma_i^s$ then $A_s = A_{s-1}$, so setting $M(\sigma) = g(A_s \upharpoonright_w)$ maintains (11).

Now suppose that $w > \gamma_i^s$. Let $n < s$. We verify (11) at stage s for n .

If $n \leq \gamma_i^{s-1}$ then $A_s \upharpoonright_n = A_{s-1} \upharpoonright_n$ and $K_s(n) = K_{s-1}(n)$, so the condition holds at stage s for n by inductive hypothesis. Now suppose that $n > \gamma_i^{s-1}$. By (10) at stage $s-1$ we have $K_{s-1}(n) > i$, and hence $K_s(n) \geq i$ (equality holds if $n = w$). Because we move the marker γ_i at stage s , we have $g(A_s \upharpoonright_n) = g(A_s \upharpoonright_w)$. Thus setting $M(\sigma) = g(A_s \upharpoonright_w)$ ensures that the condition (11) holds at stage s for n . \square

6.6. Local definition of K -triviality. The analog of the definition in Cantor space $\exists b \forall n K(x \upharpoonright_n) \leq K(0^n) + b$ should be a stronger property: From the string $x \upharpoonright_n$ we can compute the maximum distance 2^{-n} we want the highly compressible special point p to have from x . Thus, we should actually require that $K(p, n) \leq K(n) + b$ (where $K(p, n)$ is the complexity of the pair $\langle p, n \rangle$).

Definition 6.16. Let $b \in \mathbb{N}$. We say that $x \in M$ is *locally K -trivial via b* , or locally K -trivial(b) for short, if

$$(12) \quad \forall n \exists p [d(x, p) < 2^{-n} \wedge K(p, n) \leq K(n) + b].$$

K -trivial implies locally K -trivial:

Fact 6.17. *Suppose f is a K -trivial via v Cauchy name for x . Then x is locally K -trivial via $v + O(1)$.*

Proof. Note that $d(f(n), n) \leq 2^{-n}$ for each n . Clearly,

$$K(f(n), n) \leq^+ K(f \upharpoonright_{n+1}) + O(1) \leq K(n+1) + v + O(1) \leq K(n) + v + O(1).$$

Hence, for each n , the point $p = f(n)$ is a special point at a distance of at most 2^{-n} from x such that $K(p, n) \leq K(n) + v + O(1)$. \square

Showing the definition is natural. We show that the notion of locally K -trivial point in (12) is actually independent of the fact that we chose the bounds on distances to be of the form 2^{-n} . We list the positive rationals as $(r_i)_{i \in \mathbb{N}}$ in an effective way without repetitions. Note that for $\epsilon = r_i$ we have $K(\epsilon) = K(i)$ by definition. Given that, we could also define local K -triviality(b) like this:

$$(13) \quad \forall \epsilon \exists p [d(x, p) < \epsilon \wedge K(p, \epsilon) \leq K(\epsilon) + b],$$

where ϵ ranges over positive rationals. This is apparently stronger than the definition above. However, if we encode positive rationals as $(r_i)_{i \in \mathbb{N}}$ in a reasonable way then $2^{-i} \leq r_i$ for all $i \geq$ some i_0 . In this case, Fact 6.17 still holds: if n is least such that $2^{-n} \leq r_i < 2^{-n+1}$, then for $r_i = \epsilon$ we can take $p = f(n)$ as a witness for local K -triviality in the sense of (13): we have $i \geq n$, and hence

$$K(f(n), i) \leq K(f \upharpoonright_{i+1}) + O(1) \leq K(i) + v + O(1).$$

In Theorem 6.21 below we will show the converse of the foregoing Fact 6.17. Thus, being locally K -trivial, either in the original or the strong sense, is all equivalent to having a K -trivial Cauchy name.

However, this equivalence holds only for points themselves, not for their approximating sequences. The following example shows that a point x can have continuum many Cauchy names $(p_n)_{n \in \mathbb{N}}$ of special points so that (12) in Definition 6.16 holds for each n via p_n .

Example 6.18. Let the special points of the computable metric space \mathcal{M} be all the pairs $\langle r, n \rangle$, where $r \in \{0, 1\}$ and $n \in \mathbb{N}$. We declare

$$d(\langle 0, n \rangle, \langle 1, n \rangle) = 2^{-n} \text{ and } d(\langle r, n \rangle, \langle k, n+1 \rangle) = 2^{-n-1}.$$

All the special points are isolated. There is only one non-isolated point, namely $x = \lim_n \langle 0, n \rangle$. This point x is computable. Each sequence $(p_n)_{n \in \mathbb{N}}$ of the form $(\langle r_n, n \rangle)_{n \in \mathbb{N}}$ is a Cauchy name for x . For an appropriate b , (12) holds for each n via p_n .

Fact 6.19. $\#\{p \in \mathbb{N} : K(p, n) \leq K(n) + b\} = O(2^b)$.

Proof. This is like [22, 2.2.26], with the change that we let $M(\sigma) = n$ if $\mathbb{U}(\sigma) = \langle i, n \rangle$. \square

We next derive a fact on the number and distribution of the points that are locally K -trivial(b). Suppose that distinct points $x_1, \dots, x_k \in M$ are locally K -trivial(b). Pick $n^* \geq 2$ so large that $2^{-n^*+1} < d(x_i, x_k)$ for any $i \neq j$, and choose p_i for x_i, n^* according to (12). Then all the p_i are distinct. By Fact 6.19, this implies that $k = O(2^b)$. Furthermore, x_i is the only locally K -trivial(b) point x such that $d(x, p_i) \leq 2^{-n^*}$.

We summarize the preceding remarks.

Lemma 6.20.

- (a) For each $b \in \mathbb{N}$, at most $O(2^b)$ many $x \in M$ are locally K -trivial(b).

- (b) There is $n^* \in \mathbb{N}$, $n^* \geq 2$ as follows: for each point x that is locally K -trivial via b , there is a special point \tilde{p} with $K(\tilde{p}, n^*) \leq K(n^*) + b$ such that x is the only locally K -trivial(b) point within a distance of 2^{-n^*} of \tilde{p} .

We are now ready to prove the converse of Fact 6.17: local K -triviality in the sense of Definition 6.16 implies K -triviality in the sense of Definition 6.9.

Theorem 6.21. *Suppose that $x \in M$ is locally K -trivial via b . Then x has a Cauchy name h that is K -trivial via $b + O(1)$.*

Proof. After adding a constant to the Solovay function h from Fact 6.7 if necessary, we may suppose that $\forall r K(r) \leq h(r)$ and $\exists^\infty r K(r) = h(r)$. Thus, the inequalities and equalities hold literally, not merely up to a constant.

The c.e. tree T . Choose a number n^* and a special point \tilde{p} for the given point x and the constant b according to Lemma 6.20. We define a c.e. tree $T \subseteq \mathbb{N}^{<\omega}$. Since special points are identified with natural numbers, we can think of the nodes of the tree T as being labelled by special points. The root is labelled by \tilde{p} . Of course, the same label p may be used at many nodes. We think of a node at level n labelled p as the first $n + 1$ items of a Cauchy name.

As usual, $[B]$ denotes the set of (infinite) paths of a tree $B \subseteq \mathbb{N}^{<\omega}$. Each $\alpha \in [T]$ will be a Cauchy name for some point y . The special points $\alpha(i)$ will be witnesses for the local K -triviality(b) of the point y at n , where $n = n^* + i$.

Formally, we let $T = \bigcup_s T_s$, where

$$(14) \quad T_s = \{ (p_{n^*}, \dots, p_v) : p_{n^*} = \tilde{p} \wedge \forall i. n^* \leq i < v [K_s(p_i, i) \leq h(i) + b \wedge d(p_i, p_{i+1}) \leq 2^{-i-1}] \}$$

Note that $[T]$ contains a Cauchy name for x by the hypothesis that x is locally K -trivial(b), the choice of \tilde{p} and since $n^* \geq 2$. Actually, any path of T does this:

Claim 6.22. *Each $\alpha \in [T]$ is a Cauchy name of x .*

To see this, let p_0, \dots, p_{n^*-1} witness local K -triviality(b) of x for $n < n^*$. Since the metric space (M, d) is complete, α is a Cauchy name of some point $y \in M$, and the special points

$$p_0, \dots, p_{n^*-1}, \alpha(n^*), \alpha(n^* + 1), \dots$$

show that y is locally K -trivial(b). Also $d(y, \tilde{p}) \leq 2^{-n^*}$. Thus in fact $y = x$ by Lemma 6.20. This proves the claim.

For instance, in the case of the computable metric space of Example 6.18, once again let x be the only limit point. We may let $n^* = 2$ and $\tilde{p} = \langle 0, 2 \rangle$. Then T is a full binary tree, consisting of all the tuples of the form $(\tilde{p}, \langle r_3, 3 \rangle, \dots, \langle r_v, v \rangle)$ where $r_i \in \{0, 1\}$.

A very thin c.e. subtree G of T . The tree T for the computable metric space in Example 6.18 shows that there may be lots of Cauchy names with witnesses for local K -triviality; we cannot expect that each such Cauchy name is K -trivial. We will prune T to a c.e. subtree G that is so thin that all of its nodes τ are strongly compressible in the sense that $K(\tau) \leq h(|\tau|) + b + O(1)$; hence each infinite path is K -trivial by Fact 6.8.

We say that a label $p \in \mathbb{N}$ is *present at level n* of a tree $B \subseteq \mathbb{N}^{<\omega}$ if there is $\eta \in B$ such that η has length n and ends in p . While G is only a thin subtree of T , we will ensure that each label present at a level n of T is also present at level n of G . This will show that G still contains a Cauchy name for x .

To continue Example 6.18, there are only two labels at each level of T , so for $G \subseteq T$ we can simply take the tuples where each r_i is 0, except possibly the last. Then the only infinite path of G is a computable Cauchy name of the limit point x .

We will build a computable enumeration $(G_s)_{s \in \mathbb{N}}$ of the tree G where $G_s \subseteq T_s$ for each s . To help with the definition of this computable enumeration, we first define a slower computable enumeration $(\tilde{T}_s)_{s \in \mathbb{N}}$ of T that grows “one leaf at a time”. The \tilde{T}_s are subtrees of the T_s .

Why can each node of G be compressed? Suppose a new leaf labelled p appears at level n of T , but is not yet at level n of G . Suppose also that p is a successor on T of a node labelled q . Inductively, q is already on level $n-1$ of G ; that is, there already is a node $\bar{\eta}$ of length $n-1$ on G that ends in q . Since p is on level n of T , there is a \mathbb{U} -description showing that $K(p, n) \leq h(n) + b$ (that is, a string $|w|$ with $|w| \leq h(n) + b$ such that $\mathbb{U}(w) = \langle p, n \rangle$). Since p is not at level n of G , this \mathbb{U} -description is “unused”. Hence we can use it as a description of a new node $\eta = \bar{\eta} \hat{p}$ on G . This ensures that $K(\eta) \leq h(n) + b + O(1)$.

Note that we make use of the fact that a node η on G , once strongly compressible at a stage, remains so at later stages. This is why we need the Solovay function h . If we tried to satisfy the condition $K(\eta) \leq K(n) + b + O(1)$, we might fail, because $K(n)$ on the right side could decrease later on. We also needed the Solovay function to ensure that T is c.e.

In the formal construction, we build a prefix-free machine L (see [22, Ch. 2]) to give short descriptions of these nodes. The argument above is implemented via maintaining the conditions (16,17) below.

A slower computable enumeration $(\tilde{T}_s)_{s \in \mathbb{N}}$ of T . Let \tilde{T}_0 consist only of the empty string. If $s > 0$ and \tilde{T}_{s-1} has been defined, see whether there is $\tau \in T_s - \tilde{T}_{s-1}$. If so, choose τ least in some effective numbering of $\mathbb{N}^{<\omega}$. Pick v maximal such that $\tau \upharpoonright_v \in \tilde{T}_{s-1}$, and put $\tau \upharpoonright_{v+1}$ into \tilde{T}_s . Clearly we have $T = \bigcup_s \tilde{T}_s$.

Three conditions that need to be maintained at each stage. For strings $\tau, \eta \in \mathbb{N}^{<\omega}$ we write $\tau \sim \eta$ if they have the same length and end in the same elements. Recall that each label present at a level n of T needs to be also present at level n of G . Actually, in the construction we ensure that for each stage s , each label p that is present at a level n of \tilde{T}_s is also present at level n of G_s :

$$(15) \quad \forall \tau \in \tilde{T}_s \exists \eta \in G_s [\tau \sim \eta].$$

To make sure that each $\eta \in G$ satisfies $K(\eta) \leq h(|\eta|) + b + O(1)$, we construct, along with $(G_s)_{s \in \mathbb{N}}$, a computable enumeration $(L_s)_{s \in \mathbb{N}}$ of (the graph of) a prefix-free machine L . Let m, n range over natural numbers, and v, w over strings. We maintain at each stage s the conditions

$$(16) \quad \forall \eta \in G_s \forall m [0 < m \leq |\eta| \rightarrow \exists v [|v| \leq h(m) + b \wedge L_s(v) = \eta \upharpoonright_m]];$$

$$(17) \quad \text{if } \mathbb{U}_s(w) = \langle p, n \rangle, \text{ then } [w \in \text{dom}(L_s) \rightarrow p \text{ is at level } n \text{ of } G_s].$$

For full construction and verification see the paper. □

Note that all this also works for plain descriptive complexity C instead of K . We say that $x \in M$ is *locally C -trivial via b* if

$$(18) \quad \forall n \exists p [d(x, p) < 2^{-n} \wedge C(p, n) \leq C(n) + b].$$

Then by the method in the proof above but for plain machines, we have a local, “AIT” characterization of computable points in computable metric spaces:

Theorem 6.23. *A point is locally C -trivial iff it is computable.*

For details see the upcoming paper.

6.7. Incompressibility and randomness. This material is not in the paper [18].

Martin-Löf randomness is a central randomness notion. The usual definition (see [22, 3.2.1]) via passing all Martin-Löf tests works in all computable probability spaces \mathbb{I} . In particular, it can be applied both in Cantor space with the product measure, and in the unit interval with the usual uniform measure. Note that a real $r \in [0, 1]$ is Martin-Löf random iff its binary expansion is when viewed as an element of $2^{\mathbb{N}}$.

The Schnorr-Levin Theorem (see [22, 3.2.9]) states that a set $A \in 2^{\mathbb{N}}$ is Martin-Löf random if and only if there is a constant $b \in \mathbb{N}$ such that $K(A \upharpoonright_n) > n - b$ for all n .

We let

$$\begin{aligned} K(z; n) &= \min\{K(p, n) : d(z, p) < 2^{-n}\}, \\ K_*(z; n) &= \min\{K(p) : d(z, p) < 2^{-n}\}. \end{aligned}$$

Definition 6.24. We say that a point z in a computable metric space \mathcal{M} is *incompressible in approximation (i.a.)* if $\exists b \in \mathbb{N} \forall n [K(z; n) > n - b]$. We say that z is *strongly i.a.* if $\exists b \in \mathbb{N} \forall n [K_*(z; n) > n - b]$.

By the following fact, the second notion above expresses that the closer a special point is to z , the less it can be compressed.

Fact 6.25. z is strongly i.a. via $b \Leftrightarrow \forall p [d(z, p) \geq 2^{-K(p)-b}]$.

Proof. Note that the condition $\forall p [d(z, p) \geq 2^{-K(p)-b}]$ is equivalent to

$$\forall n [d(z, p) < 2^{-n} \rightarrow K(p) + b > n],$$

i.e., $\forall n K_*(z; n) > n - b$. □

Fact 6.26. Let \mathcal{M} be either Cantor space or the unit interval, with the computable structures introduced in Example 6.4. Let x be an element of \mathcal{M} . Then the following are equivalent.

- (i) x is incompressible in approximation.
- (ii) x is strongly incompressible in approximation.
- (iii) x is Martin-Löf random.

Proof. (i) \rightarrow (iii). We first consider the case that x is a bit sequence A in Cantor space. Suppose x is incompressible in approximation via $b \in \mathbb{N}$.

Given $n \in \mathbb{N}$, let p be the special point $A \upharpoonright_n 0^\infty$, that is, p consists of the first n bits of A , followed by 0s. Then, since $d(A, p) < 2^{-n}$, by our hypothesis we have $K(A \upharpoonright_n) \geq^+ K(p, n) > n - b$. This shows that A is Martin-Löf random.

Now suppose $x \in [0, 1]$ is incompressible in approximation. Then $x < 1$. Let A be the binary expansion of x with infinitely many 0s. If p is a special point in $2^{\mathbb{N}}$ with $d(A, p) < 2^{-n}$, then the dyadic rational q corresponding to p is a special point in $[0, 1]$ with $d(x, q) < 2^{-n}$. Since q can be computed from p , this shows that A is incompressible in approximation within Cantor space. Hence x is Martin-Löf random.

(iii) \rightarrow (ii). The following argument works in both Cantor space and the unit interval. Let

$$U_b = \bigcup_p B(p, 2^{-K(p)-b-1}).$$

Then the sequence $(U_b)_{b \in \mathbb{N}}$ of open sets is uniformly c.e. Furthermore, the measure of U_b is bounded by

$$\sum_p 2^{-K(p)-b} \leq 2^{-b} \Omega.$$

Thus, $\{U_b\}_{b \in \mathbb{N}}$ is a Martin-Löf test. Since z passes this test, z is incompressible in approximation. \square

This only works because we have dimension 1. In the unit *square*, for instance, a ML-random x will satisfy $K_*(x; n) \geq^+ 2n$, by a proof similar as above. In the k -cube you guessed it, $\geq^+ kn$. In these spaces there is a computable upper bound on $K_*(x; n)$. This fails for instance in $\mathcal{C}[0, 1]$

We give an example of a computable metric spaces such that no incompressible point exists.

Example 6.27. Let \mathcal{M} be the Cantor space $2^{\mathbb{N}}$ with the distance function squared, i.e., with $\tilde{d}(f, g) = d(f, g)^2$, and the same computable structure as in Example 6.4(iii). Then no point in \mathcal{M} is incompressible in approximation.

To see this, suppose that $A \in \mathcal{M}$ is incompressible in approximation via b . Let p_n be the special point $(A \upharpoonright_n)0^\infty$. Then $\tilde{d}(A, p_n) < 2^{-2n}$, whence $K(p_n, n) \geq K(A; 2n) > 2n - b$. For large n , this contradicts the fact that $K(p_n, n) \leq^+ K(A \upharpoonright_n) \leq^+ n + 2 \log n$.

We show that being i.a. is preserved under any computable map $F: \mathcal{M} \rightarrow \mathcal{N}$ such that F^{-1} is Lipschitz. For instance, via a computable embedding of the unit interval we can find points that are i.a. in any computable Banach space.

Proposition 6.28. *Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a computable map such that F^{-1} is Lipschitz, namely, there is $v \in \mathbb{N}$ such that $d(x, y) \leq 2^v d(F(x), F(y))$. Then for each z in \mathcal{M} and each $n \geq v + 1$ we have*

$$K(F(z); n) \geq^+ K(z; n - v - 1).$$

Proof. Define a prefix-free machine L as follows. On input τ , if $\mathbb{U}(\tau) = \langle q, n \rangle$ for a special point q of \mathcal{N} , then L searches for a special point p of \mathcal{M} such that $d(F(p), q) < 2^{-n}$. If p is found, it outputs $\langle p, n - v - 1 \rangle$.

Suppose now that $\mathbb{U}(\tau) = \langle q, n \rangle$ for some τ of least length, where

$$d(F(z), q) < 2^{-n}.$$

Then on input τ the machine L finds p because F is continuous. This shows that $K(p, n - v - 1) \leq^+ |\tau| = K(q, n)$. Furthermore, we have

$$d(p, z) \leq 2^v d(F(p), F(z)) \leq 2^v [d(F(p), q) + d(q, F(z))] < 2^v 2^{-n+1}.$$

Since $K(F(z); n)$ is the minimum of all such $K(q, n)$ such that $d(F(z), q) < 2^{-n}$, this establishes the required inequality. \square

In particular, F preserves being i.a. To preserve being strongly i.a. we also need that the range of F is dense:

Proposition 6.29. *Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a computable map with range dense in \mathcal{N} such that F^{-1} is Lipschitz with constant 2^v . If z in \mathcal{M} is strongly i.a. then so is $F(z)$. In particular, being strongly i.a. is preserved under changing the computable structure to an equivalent one.*

Proof. Suppose z is strongly i.a. via b . Then, by Fact 6.25, for each special point p of \mathcal{M} and each σ , we have

$$(19) \quad \mathbb{U}(\sigma) = p \rightarrow d(z, p) \geq 2^{-|\sigma| - b}.$$

We define a prefix-free machine L . By the recursion theorem, we may assume that a coding constant d_L for L is given in advance. On input τ , if $\mathbb{U}(\tau) = q$ for a special point q of \mathcal{N} , then L outputs a special point p of \mathcal{M} such that

$$d(F(p), q) < 2^{-|\tau| - v - b - d_L - 1}.$$

Note that p exists because the range of F is dense in \mathcal{N} .

Since d_L is the coding constant for L , we have $\mathbb{U}(\sigma) = p$ for some σ such that $|\sigma| \leq |\tau| + d_L$. Thus, by (19) $d(z, p) \geq 2^{-|\sigma|-b} \geq 2^{-|\tau|-d_L-b}$. By the Lipschitz condition on F^{-1} we obtain

$$\begin{aligned} d(F(z), q) &\geq d(F(z), F(p)) - d(F(p), q) \\ &\geq 2^{-v} d(z, p) - d(F(p), q) \\ &\geq 2^{-|\tau|-v-d_L-b} - 2^{-|\tau|-v-d_L-b-1} \\ &= 2^{-|\tau|-v-d_L-b-1}. \end{aligned}$$

If $|\tau| = K(q)$ then by Fact 6.25, this shows that $F(z)$ is strongly i.a. in \mathcal{N} via the constant $v + d_L + b + 1$. \square

Part 4. Various**7. SOME NEW EXERCISES ON COMPUTABILITY AND RANDOMNESS**

These will go into next edition of Nies' book. Solutions next page.

Stephan showed that every wtt-incomplete c.e. set B is "i.o. K -trivial" in the sense that $\exists^\infty n K(B \upharpoonright_n) \leq^+ K(n)$. See [22, 5.2.8]. The following exercise shows that the weakly K -trivials exist in every wtt degree.

Fact 7.1. *For each set A there is a set $B \equiv_{\text{wtt}} A$ such that*

$$\exists^\infty n K(B \upharpoonright_n) \leq^+ K(n).$$

Moreover the set of n where this happens is computably bounded.

We consider the smallest cost function that makes sense at all. Ω is random and hence does not obey even that.

Fact 7.2. *Let $c(x, s) = 2^{-x}$. Show that Ω does not obey this cost function.*

Fact 7.3. *(Yu Liang) Use Arslanov's completeness criterion to give a direct proof that Ω_0 is low.*

Solutions:

7.1 Define sets $(B_k)_{k \geq -1}$ of size $k+1$ inductively (but not computably), as follows. Let $B_{-1} = \emptyset$. If B_k has been defined, let $n = n_k$ be the number such that $D_n = B_k$ (strong index). Note that $n_k > \max B_k$. If $k \in A$ let $B_{k+1} = B_k \cup \{2n\}$. Otherwise, let $B_{k+1} = B_k \cup \{2n+1\}$.

Verification. Clearly $B \leq_{\text{wtt}} A$ by the recursive definition of B . We also have $A \leq_{\text{wtt}} B$ because the sequence (n_k) is computably bounded.

To show that B is i.o. K -trivial, note that for $m = 2n_k$, we have that $K(B \upharpoonright_m) \leq^+ K(m)$ because $D_{n_k} = B \cap [0, m)$.

7.2 We view Ω_s as a binary string. At stage $s > 0$, if there is i least such that $\Omega_s(i) = 1$ and $\Omega_{s-1}(i) = 0$, enumerate the interval $[\Omega_s \upharpoonright_{i+1}]$ into a set \mathcal{S} .

If Ω obeys c then \mathcal{S} is a Solovay test capturing Ω .

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